

Generalized Hitchin systems on rational surfaces

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Abstract

By analogy with work of Hitchin on integrable systems, we construct natural relaxations of several kinds of moduli spaces of difference equations, with special attention to a particular class of difference equations on an elliptic curve (arising in the theory of elliptic special functions). The common feature of the relaxations is that they can be identified with moduli spaces of sheaves on rational surfaces. Not only does this make various natural questions become purely geometric (rigid equations correspond to -2 -curves), it also establishes a number of nontrivial correspondences between different moduli spaces, since a given moduli space of sheaves is typically the relaxation of infinitely many moduli spaces of equations. In the process of understanding this, we also consider a number of purely geometric questions about rational surfaces with anticanonical curves; e.g., we give an essentially combinatorial algorithm for testing whether a given divisor is the class of a -2 -curve or is effective with generically integral representative.

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1 Introduction

One of the more striking properties of hypergeometric functions is that they are the solutions of a *rigid* differential equation, i.e., one which is determined by its order and its singularities. For instance, it was already observed by Riemann that any second-order Fuchsian differential equation with exactly three singular points reduces by a simple change of variables to the equation satisfied by a hypergeometric function of type ${}_2F_1$. More generally, the equation satisfied by a hypergeometric function of type ${}_rF_{r-1}$ is rigid for any r ; it is r -th order, with Fuchsian singularities at 0, 1, and ∞ , and is uniquely determined by the exponents at 0 and 1, together with a more significant constraint on the singularity at ∞ . This suggests that one should study rigid equations more generally; for an exploration of this from the monodromy perspective, see [27]. In addition, the modern theory of Painlevé transcendents leads us to consider what happens for non-rigid equations; for instance, the Painlevé VI equation can be interpreted as a flow in a 2-dimensional moduli space of second order Fuchsian equations with four singular points, with specified exponents at the singular points.

If we extend our focus to include q -hypergeometric functions, then we see that we must consider more than just differential equations: Gauss' differential equation becomes a q -difference equation when extended to q -hypergeometric functions. Even more generally, we could consider *elliptic* hypergeometric functions¹, and thus elliptic difference equations.

If we look carefully at the equations satisfied by most of the known elliptic hypergeometric functions [47], or the equations satisfied by semiclassical elliptic biorthogonal functions [46], we find that there is an important additional structure. A general elliptic difference equation has the form

$$v(z + q) = A(z)v(z) \tag{1.1}$$

where q is a point of an elliptic curve and $A(z)$ is a matrix of elliptic functions (with $\det A(z)$ not identically 0); in the cases of interest, though, we have the additional constraint

$$A(-q - z) = A(z)^{-1}. \tag{1.2}$$

This is more natural than it may appear. We can view a difference equation as a 1-cocycle for the group \mathbb{Z} acting on $\mathrm{GL}_n(k(E))$ via translation by q , and cohomologous 1-cocycles are simply related by gauge transformations (called isomonodromy transformations in [46], as they preserve a suitable notion of monodromy)

$$A(z) \mapsto C(z + q)A(z)C(z)^{-1}; \tag{1.3}$$

i.e., the equation satisfied by $C(z)v(z)$ is cohomologous to the equation satisfied by $v(z)$. In the cases with symmetric equations, the solutions of interest are symmetrical: they satisfy the additional constraint $v(-z) = v(z)$. Now, the *pair* of equations

$$v(z + q) = A(z)v(z), \quad v(-z) = v(z) \tag{1.4}$$

can also be viewed as an object in nonabelian cohomology, namely a 1-cochain for the infinite dihedral group. The constraint on A simply says that this cochain is a cocycle, basically a formal

¹Very roughly speaking, these are series in which the usual hypergeometric constraint “ratios of consecutive terms are rational functions of the index” is replaced by “ratios of consecutive terms are *elliptic* functions of the index”; we will not need details, as we are considering these for motivation only.

self-consistency condition. Indeed, a symmetrical solution of the difference equation satisfies

$$v(-q-z) = A(z)v(z) \quad (1.5)$$

and thus

$$v(z) = A(-q-z)v(-q-z) = A(-q-z)A(z)v(z). \quad (1.6)$$

If $A(-q-z) \neq A(z)^{-1}$, then we will have “too few” symmetric solutions. (Note that two cocycles for the infinite dihedral group are cohomologous iff they are related by a gauge transformation with $C(-z) = C(z)$.)

We are thus led to the following natural question: What are the *symmetric* elliptic difference equations which are rigid? In particular, we would expect, and will see, that the elliptic hypergeometric equations are indeed rigid, and the equations related to the elliptic Painlevé equation fit into a 2-dimensional moduli space. This, of course, degenerates to corresponding questions about symmetric ordinary and q -difference equations, and at those levels can also be degenerated to questions about ordinary and q -difference equations without symmetry, not to mention differential equations on \mathbb{P}^1 .

In the case of differential equations, there is a useful relaxation of the problem due to Hitchin [21]. Rather than consider differential equations themselves, i.e., connections on vector bundles over \mathbb{P}^1 , one considers 1-form valued endomorphisms $V \rightarrow V \otimes \Omega$. When $V \cong \mathcal{O}_{\mathbb{P}^1}^n$ is a trivial bundle, these notions are essentially the same: a 1-form valued endomorphism A corresponds to a differential equation $dv = Av$. While the notions diverge for nontrivial bundles, we can still expect that a large open subset of the two moduli spaces should coincide. There is a corresponding relaxation for difference equations [23]; again, we classify matrices rather than difference equations, but the two moduli problems are closely related.

A significant advantage of the relaxation over the original problem is that it reduces to a moduli problem about sheaves on a smooth projective surface. The relaxed version of a differential equation on a smooth curve C corresponds to a sheaf on the ruled surface $\mathbb{P}(\mathcal{O}_C \oplus \omega_C)$, while the relaxed version of an elliptic difference equation (without symmetry) corresponds to a sheaf on $E \times \mathbb{P}^1$ (in either case, this is generically a line bundle on a certain spectral curve). The information about singularities translates to a specification of how said sheaf meets a certain anticanonical curve on the surface (the zero locus of a Poisson structure).

In the case of differential equations on \mathbb{P}^1 , the surface in question is the Hirzebruch surface $F_2 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$, and the relevant anticanonical curve has the form $2S$ where S is a section disjoint from the -2 curve on F_2 . In contrast to the case of ruled surfaces of higher genus, this anticanonical curve is extremely special, and one might thus wonder whether there is a natural interpretation for the moduli spaces associated to more general anticanonical curves on F_2 . The most general anticanonical curve on F_2 is in fact a smooth curve of genus 1 (more precisely, a *hyperelliptic* curve of genus 1; specifying an embedding in F_2 is equivalent to specifying a degree 2 map to \mathbb{P}^1), and as we will see below, the corresponding moduli problem is a relaxation of the moduli problem of symmetric elliptic difference equations.

The fact that symmetric elliptic difference equations correspond to sheaves on a *rational* surface appears to be at the core of why they appear in special function theory. Indeed, as we noted in [42], rigid sheaves in the relaxation can only exist on a rational surface (even the trivial equation $v(z+q) = v(z)$ fails to be rigid as a nonsymmetric elliptic difference equation!). In addition, the birational maps between irrational ruled surfaces are extremely simple (between minimal surfaces, all birational maps are compositions of elementary transformations), while rational surfaces have a rich structure coming from birational maps. As we will see, this means that any given sheaf actually

corresponds to a large (in some cases infinite) set of inequivalent equations; in higher genus cases, all we can do is multiply v by the solution of a first-order equation.

The purpose of the present note is to explore these relaxations, and in particular the additional structure afforded by the fact that they live on a rational surface. We first show how to translate a symmetric elliptic difference equation into a sheaf on a rational surface (including mild generalizations where we twist by a line bundle), and discuss how this degenerates to ordinary and q -difference cases. Note that there is a somewhat subtle issue here, in that what it means to be singular changes slightly if we forget the symmetry of the equation (consider the equation $v(z + q) = -v(z)$, which has no *symmetric* solutions which are holomorphic and nonzero near $z = -q/2$). This is one reason why we should indeed think of symmetric elliptic equations as more than just a special case of elliptic equations. We also consider a few other moduli problems that also reduce to questions about sheaves on rational surfaces.

Given this translation, considerations of [42] reduce questions of rigidity to much simpler questions in algebraic geometry. Indeed, the sheaves corresponding to relaxations of rigid difference/differential equations are just direct images of line bundles on -2 curves on suitable blowups of the original ambient surface (specifically, -2 curves which are disjoint from the anticanonical curve). Similarly, the 2-dimensional moduli spaces (there is an induced symplectic structure, so all moduli spaces here are even-dimensional) are related to (quasi-)elliptic pencils.

In this way, our questions about moduli problems of difference equations translate to structural questions about rational surfaces with an anticanonical curve: what are the -2 curves, and which divisor classes have integral representatives disjoint from the anticanonical curve? And, of course, what are the different ways of blowing a given rational surface down to a Hirzebruch surface? Earlier work on blowups of \mathbb{P}^2 leads us to a certain family of Coxeter groups, which almost acts on the set of ways of blowing down; each simple reflection acts unless a corresponding divisor class is effective. Using this action, we obtain algorithms for determining (a) whether a given divisor class is the class of a -2 curve (i.e., whether the corresponding sheaves represent rigid equations), and (b) whether a given divisor class is effective (or nef, or integral).

The one major drawback of the relaxation is that the various natural transformations of sheaves (changing the blowdown, twisting by a line bundle on a blowup) will almost always act in the wrong way from the difference equation perspective. (E.g., twisting has the effect of conjugating the matrix A by a suitable rational matrix, and this needs to be replaced by a suitable q -deformed conjugation.) Since the relaxation lives on a *Poisson* rational surface, it is natural to conjecture that the original problem should correspond to sheaves on a *noncommutative* rational surface. This is bolstered by recent work [37] showing that one can obtain the elliptic Painlevé equation as a Hitchin-type system on a noncommutative \mathbb{P}^2 . In a future paper [44], we will show how to use elliptic difference operators to construct a suitable family of noncommutative rational surfaces, and extend this to general ruled surfaces in [41]; this will require some additional facts about commutative rational surfaces which we establish here. In particular, our noncommutative rational surfaces will be constructed via certain flat families of difference operators, and it is already a nontrivial fact, established below, that the corresponding spaces are flat in the commutative setting.

Related to this, we also consider some general questions about the moduli space (stack) of anticanonical rational surfaces. In particular, this moduli stack naturally splits as a union of locally closed substacks based on the structure of the anticanonical curve. This leads to the question of how this different pieces are related, specifically how their closures intersect. This appears to be a rather hard problem in general; we give an easy necessary condition for one such substack to be contained in the closure of another, as well as a much more subtle necessary condition, which leads to some pathologies in small characteristic. In particular, we give a corrected version of the diagram of degenerations of surfaces with $K^2 = 0$ (corresponding to Sakai's hierarchy [50] of discrete and

continuous Painlevé equations). We also discuss in detail how the structure of the anticanonical curve relates to the structure of the corresponding difference/differential equations.

We will then conclude with a couple of sections discussing the implications of these results for symmetric elliptic difference equations (including what most of the natural operations do both in the relaxed and in the nonrelaxed versions), as well as certain degenerate cases. The latter include natural birational maps between spaces of symmetric q -difference equations and spaces of nonsymmetric q -difference equations, as well as maps between such equations and solutions of the “multiplicative Deligne-Simpson problem”. This includes settling a conjecture of [15], as a special case of a theorem identifying the Jacobian of a rational elliptic surface (Theorem 7.1 below). We also briefly consider some deformations generalizing certain Calogero-Moser spaces.

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2 Sheaves from difference equations

As we discussed in the introduction, the analogue of a differential equation at the top (elliptic) level in the hierarchy of special functions is a *symmetric elliptic difference equation*, which we should think of as the pair of equations

$$v(z + q) = A(z)v(z), \quad v(-z) = v(z), \quad (2.1)$$

where A is a matrix of elliptic functions subject to the consistency condition $A(-q - z)A(z) = 1$. The natural relaxation of this problem is to forget the difference equation, and simply classify matrices A of elliptic functions satisfying $A(-q - z)A(z) = 1$. (We will also want to take into account singularities, but will table that question for the moment.)

Since we plan to relate this to an algebraic geometric object, it will be helpful to rephrase this original problem in a somewhat more abstractly geometric way. Thus we suppose given a smooth genus 1 curve C_α over an algebraically closed field k (not necessarily of characteristic 0), along with a translation $\tau_q : C_\alpha \rightarrow C_\alpha$ and a hyperelliptic involution $\eta : C_\alpha \rightarrow C_\alpha$, i.e., such that the quotient of C_α by the involution is isomorphic to \mathbb{P}^1 . In the analytic setting, C_α is \mathbb{C}/Λ for some lattice Λ , τ_q is the map $z \mapsto z + q$, and η is the map $z \mapsto -q - z$. We take the latter choice for η so that the problem of classifying A becomes the following: Classify matrices $A \in \text{GL}_n(k(C_\alpha))$ such that $\eta^*A = A^{-1}$.

Just as the original problem can be rephrased in terms of 1-cocycles of the infinite dihedral group on $\text{GL}_n(k(C_\alpha))$, this question is itself related to nonabelian cohomology: a matrix A such that $\eta^*A = A^{-1}$ specifies a 1-cocycle for the cyclic group $\langle \eta \rangle$ (of order 2). Now, the action of η allows us to think of $k(C_\alpha)$ as a Galois extension of the invariant subfield $k(\mathbb{P}^1)$, and thus we find

$$H^1(\langle \eta \rangle; \text{GL}_n(k(C_\alpha))) = H^1(\text{Gal}(k(C_\alpha)/k(\mathbb{P}^1)), \text{GL}_n). \quad (2.2)$$

It is a classical fact that the latter Galois cohomology set is trivial, and this translates to the following fact (often referred to as Hilbert’s Theorem 90, though Hilbert only considered the case of a cyclic Galois group acting on GL_1).

Proposition 2.1. *Let L/K be a quadratic field extension, and let $A \in \text{GL}_n(L)$ be a matrix such that $\bar{A} = A^{-1}$, where $\bar{\cdot}$ is the conjugation of L over K . Then there exists a matrix $B \in \text{GL}_n(L)$ such that $A = \bar{B}B^{-1}$, and B is unique up to right-multiplication by $\text{GL}_n(K)$.*

Proof. In this case, the argument is particularly simple. Given any vector $w \in L^n$, the vector $v = \bar{w} + A^{-1}w$ satisfies $\bar{v} = Av$. If we apply this to a basis of L^n over K , we obtain in this way at least n vectors satisfying $\bar{v} = Av$ which are linearly independent over K . It follows that there exists a matrix $B \in \mathrm{GL}_n(L)$ such that $\bar{B} = AB$, which is what we want. If B' is another such matrix, then

$$\overline{B^{-1}B'} = B^{-1}A^{-1}AB' = B^{-1}B', \quad (2.3)$$

and thus $B^{-1}B' \in \mathrm{GL}_n(K)$ as required. \square

In our setting, it will turn out to be appropriate to make the factorization have the form $A = \eta^* B^{-t} B^t$. (In the noncommutative setting, the most natural correspondence between difference equations and sheaves is contravariant and holomorphic in B .) The nonuniqueness (we can still multiply B on the right by any element of GL_n) is of course still an issue, but it turns out there is a slight modification which can be made unique. The first step is to make the nonuniqueness problem worse by allowing B to be a map between vector bundles. Let $\pi_\eta : C_\alpha \rightarrow \mathbb{P}^1$ be the morphism quotienting by the action of η . Then for any vector bundle V on \mathbb{P}^1 , and any meromorphic (and generically invertible) map

$$B : \pi_\eta^* V \rightarrow \mathcal{O}_{C_\alpha}^n, \quad (2.4)$$

we obtain a well-defined matrix $\eta^* B^{-t} B^t$, and of course any matrix with $\eta^* A = A^{-1}$ can be represented in this way (just take V to be $\mathcal{O}_{\mathbb{P}^1}^n$). (Here, by the transpose B^t , we mean the image of B under the functor $\mathrm{Hom}_{C_\alpha}(-, \mathcal{O}_{C_\alpha})$.) The advantage of allowing V to be a more general vector bundle is that we can then insist that B be *holomorphic* (and thus injective), by absorbing any poles into V . This is still non-unique, since we could freely replace V by any vector bundle it contains, and still obtain an injective morphism supporting a factorization of A . However, we can make this unique by imposing a maximality condition on V .

Proposition 2.2. *Suppose $B : \pi_\eta^* V_0 \rightarrow k(C_\alpha)^n$ is an injective map of sheaves, with V_0 a rank n vector bundle on \mathbb{P}^1 . This induces an isomorphism $B : \pi_\eta^*(V_0 \otimes_{\mathcal{O}_{\mathbb{P}^1}} k(\mathbb{P}^1)) \cong k(C_\alpha)^n$, and the set of bundles $V \subset V_0 \otimes_{\mathcal{O}_{\mathbb{P}^1}} k(\mathbb{P}^1)$ such that $BV \subset \mathcal{O}_{C_\alpha}^n$ is nonempty, with a unique maximal element.*

Proof. That the induced map of vector spaces over $k(C_\alpha)$ is an isomorphism follows from the fact that it is an injective map of vector spaces of the same dimension. That the set of bundles V is nonempty is straightforward, as we have already mentioned (just absorb any poles of B into V). Finally, if V_1, V_2 are vector bundles contained in $V_0 \otimes_{\mathcal{O}_{\mathbb{P}^1}} k(\mathbb{P}^1)$ such that $BV_1, BV_2 \subset \mathcal{O}_{C_\alpha}^n$, then $V_1 + V_2$ is still contained in $V_0 \otimes_{\mathcal{O}_{\mathbb{P}^1}} k(\mathbb{P}^1)$, so is torsion-free, thus a vector bundle; and $B(V_1 + V_2) = BV_1 + BV_2 \subset \mathcal{O}_{C_\alpha}^n$. Since $BV \subset \mathcal{O}_{C_\alpha}^n$ implies $\deg(\pi_\eta^* V) = \deg(V) \leq 0$, it follows that there is a unique maximal such bundle. \square

To summarize the above considerations, given any matrix $A \in \mathrm{GL}_n(k(C_\alpha))$ such that $\eta^* A = A^{-1}$, there is a canonical factorization $A = \eta^* B^{-t} B^t$ where B is an injective morphism

$$B : \pi_\eta^* V \rightarrow \mathcal{O}_{C_\alpha}^n \quad (2.5)$$

with V a rank n vector bundle on \mathbb{P}^1 , maximal among those supporting a map B . This canonical factorization also clarifies issues regarding singularities. For instance, as we mentioned in the introduction, the equation $v(z + q) = -v(z)$ is singular at $-q/2$ as a symmetric equation, since there are no symmetric solutions which are nonzero and holomorphic at $-q/2$. This is nonobvious in terms of A , since $A = -1$ has no zeros or poles here, but becomes clear in terms of B , as we find that B must in fact vanish at every point of the form $-q/2$ (more precisely, at every fixed point of η). We find in general that the points where $\tau_q^* v = Av$ is singular as a symmetric equation are

precisely those points where $\det(B) = 0$. More generally, the right way to classify singularities of (symmetric) elliptic difference equations is to consider the induced cocycle over the ring of adèles; one can show that the classes of such cocycles are determined by the corresponding elementary divisors of B .

Remark. A similar factorization appeared in [46], but the reader should be cautioned that they are not quite the same; indeed, the factorization of [46] involves a partition of the singularities in to two subsets, and depends significantly on that choice. It turns out that those matrices (up to transpose) correspond to canonical factorizations of cohomologous equations, see Section 8.

Since B is a map from a pullback, we can use adjunction to relate it to a map to a direct image: specifying B is equivalent to specifying

$$\pi_{\eta*}B : V \rightarrow \pi_{\eta*}\mathcal{O}_{C_\alpha}^n. \quad (2.6)$$

With this in mind, we can obtain a natural extension of B to a surface containing C_α (as an anticanonical curve). Indeed, since π_η has degree 2, the direct image $\pi_{\eta*}\mathcal{O}_{C_\alpha}$ is a vector bundle of degree 2, and thus we can take the corresponding projective bundle to obtain a Hirzebruch surface $X = \mathbb{P}(\pi_{\eta*}\mathcal{O}_{C_\alpha})$. Note that $X \cong F_2$, since $\pi_{\eta*}\mathcal{O}_{C_\alpha} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. Moreover, X contains C_α in a natural way, in such a way that the induced map from C_α to \mathbb{P}^1 is just π_η , and C_α is anticanonical. If $\rho : X \rightarrow \mathbb{P}^1$ is the corresponding ruling, and s_{\min} denotes the section of the ruling with minimal self-intersection ($s_{\min}^2 = -2$), then we have a canonical isomorphism

$$\pi_{\eta*}\mathcal{O}_{C_\alpha} \cong \rho_*\mathcal{L}(s_{\min}), \quad (2.7)$$

since $\mathcal{L}(s_{\min})$ is the relative $\mathcal{O}(1)$. This is just the direct image under ρ of the restriction map

$$\mathcal{L}(s_{\min}) \rightarrow \mathcal{L}(s_{\min})|_{C_\alpha} \cong \mathcal{O}_{C_\alpha}, \quad (2.8)$$

where we note that C_α and s_{\min} are disjoint, and we make the isomorphism canonical by taking the unique global section of $\mathcal{L}(s_{\min})$ to the unique global section of \mathcal{O}_{C_α} .

In other words, to specify $B : \pi_\eta^*V \rightarrow \mathcal{O}_{C_\alpha}^n$, it is equivalent to specify its direct image

$$\rho_*B : V \rightarrow \rho_*\mathcal{L}(s_{\min})^n, \quad (2.9)$$

where we now think of B as a morphism of sheaves on $C_\alpha \subset X$. Again using the adjunction between ρ_* and ρ^* gives us a morphism

$$B : \rho^*V \rightarrow \mathcal{L}(s_{\min})^n, \quad (2.10)$$

which when restricted to $C_\alpha \subset X$ recovers the original morphism. Again, we modify this slightly to

$$B : \rho^*V \otimes \mathcal{L}(-s_{\min}) \rightarrow \mathcal{O}_X^n, \quad (2.11)$$

which has no effect on the restriction to C_α , but is more natural in the noncommutative setting (and slightly more natural even in the commutative setting). In any event, we now have a morphism of vector bundles on the Hirzebruch surface X . This in turn translates to a questions about sheaves, via the following result.

Proposition 2.3. *Let $\rho : X \rightarrow C$ be a ruled surface, with relative $\mathcal{O}(1)$ denoted by $\mathcal{O}_\rho(1)$, and let M be a coherent sheaf on X . Then the following are equivalent.*

1. *M is the cokernel of an injective morphism*

$$B : \rho^*V \otimes \mathcal{O}_\rho(-1) \rightarrow \rho^*W \quad (2.12)$$

with V, W vector bundles of the same rank on C .

2. M has 1-dimensional support, $M \otimes \mathcal{O}_\rho(-1)$ is ρ_* -acyclic, and ρ_*M is torsion-free.

Moreover, if either condition holds, then B is uniquely determined up to isomorphism by M .

Proof. 1 \implies 2: Since B is an injective morphism of vector bundles of the same rank, it is an isomorphism on the generic fiber, and thus $\text{supp}(M)$ does not contain the generic point of X . It follows that M has ≤ 1 -dimensional support. (In fact, M is supported on the zero locus of $\det(B)$.)

Now, since the sheaves $\mathcal{O}_\rho(d)$ are isomorphic to $\mathcal{O}_f(d)$ on every fiber f , we find that $\mathcal{O}_\rho(d)$ is ρ_* -acyclic for $d \geq -1$, and has trivial direct image for $d \leq -1$. In particular, we can compute the higher direct image long exact sequence associated to the short exact sequence

$$0 \rightarrow \rho^*V \otimes \mathcal{O}_\rho(-2) \rightarrow \rho^*W \otimes \mathcal{O}_\rho(-1) \rightarrow M \otimes \mathcal{O}_\rho(-1) \rightarrow 0. \quad (2.13)$$

Since ρ has 1-dimensional fibers, this long exact sequence terminates after degree 1, and we conclude that $M \otimes \mathcal{O}_\rho(-1)$ is ρ_* -acyclic. Similarly, from the untwisted short exact sequence, we obtain

$$W \cong \rho_*\rho^*W \cong \rho_*M. \quad (2.14)$$

In particular, ρ_*M is torsion-free, and we can recover B as the kernel of the natural map $\rho^*\rho_*M \rightarrow M$.

2 \implies 1: The condition that $M \otimes \mathcal{O}_\rho(-1)$ is ρ_* -acyclic implies that if we view M as a family of sheaves on \mathbb{P}^1 , then every fiber satisfies $H^1(M_f(-1)) = 0$. In particular, every fiber is 0-regular in the sense of Castelnuovo and Mumford, and thus M is relatively globally generated [29]. Since ρ_*M is torsion-free by assumption, so a vector bundle, it remains only to show that the kernel of this natural map has the form $\rho^*V \otimes \mathcal{O}_\rho(-1)$. Now, M cannot have any 0-dimensional subsheaf, since that would produce a 0-dimensional subsheaf of ρ_*M . In other words, M is a pure 1-dimensional sheaf, and thus has homological dimension 1. In particular, the kernel is a vector bundle (of the same rank as W , since the map is generically surjective), so we can view it as a flat family of sheaves on \mathbb{P}^1 . Since $\rho^*\rho_*M$ and M are acyclic with isomorphic direct image, it follows that the kernel has trivial direct image and higher direct image, and thus every fiber of the kernel has trivial cohomology. The only sheaves on \mathbb{P}^1 with trivial cohomology are sums of $\mathcal{O}_{\mathbb{P}^1}(-1)$, and thus the kernel has the form

$$V' \otimes \mathcal{O}_\rho(-1) \quad (2.15)$$

where V' is a flat family of sheaves on \mathbb{P}^1 , each fiber of which is a power of $\mathcal{O}_{\mathbb{P}^1}$. In other words, $V' \cong \rho^*V$ for some vector bundle V . \square

Remark 1. Just as we found $W \cong \rho_*M$, we can also compute V from M , since

$$\rho_*(M \otimes \mathcal{O}_\rho(-1)) \cong V \otimes R^1\rho_*\mathcal{O}_\rho(-2), \quad (2.16)$$

and $R^1\rho_*\mathcal{O}_\rho(-2)$ is a line bundle on C .

Remark 2. This argument was inspired by the main construction of [7], which considered minimal resolutions of sheaves on \mathbb{P}^n for $n > 1$; in our case, we have a *relative* minimal resolution of a family of sheaves on \mathbb{P}^1 .

Of course, there remain two conditions to translate into conditions on the sheaf M , namely the constraint on the singularities, and the constraint that V is maximal. The former is straightforward: specifying the elementary divisors of B along C_α is equivalent to specifying the cokernel of B as a morphism of vector bundles on C_α , and thus the singularities are determined by the restriction $M|_{C_\alpha}$. (In particular, we have the overall constraint that M must be transverse to C_α , so that B is generically invertible on C_α !) The latter is somewhat more subtle, but is not too difficult to deal with.

Proposition 2.4. *Let $\rho : X \rightarrow C$ be a ruled surface, and suppose the sheaf M is given by a presentation*

$$0 \rightarrow \rho^*V \otimes \mathcal{O}_\rho(-1) \xrightarrow{B} \rho^*W \rightarrow M \rightarrow 0, \quad (2.17)$$

where V and W are vector bundles of the same rank. The morphism B extends to a supersheaf $V \subsetneq V' \subset V \otimes_{\mathcal{O}_C} k(C)$ iff M has a subsheaf of the form $\mathcal{O}_f(-1)$ for some fiber f of ρ .

Proof. If B extends to V' , then the image of $\rho^*V' \otimes \mathcal{O}_\rho(-1)$ induces a subsheaf of M isomorphic to

$$\rho^*(V'/V) \otimes \mathcal{O}_\rho(-1). \quad (2.18)$$

Now, V'/V is 0-dimensional, so contains a subsheaf of the form \mathcal{O}_p for some closed point $p \in C$. This \mathcal{O}_p itself induces a supersheaf of V , and thus a subsheaf of M of the form

$$\rho^*(\mathcal{O}_p) \otimes \mathcal{O}_\rho(-1) \cong \mathcal{O}_f(-1), \quad (2.19)$$

where f is the fiber over p .

Conversely, suppose we have an injective map $\mathcal{O}_f(-1) \rightarrow M$, and let M' be the cokernel. The higher direct image long exact sequences tell us

$$\rho_*M' \cong \rho_*M \quad R^1\rho_*M' \cong R^1\rho_*M = 0 \quad R^1\rho_*(M' \otimes \mathcal{O}_\rho(-1)) \cong R^1\rho_*(M' \otimes \mathcal{O}_\rho(-1)) = 0, \quad (2.20)$$

and thus M' has a presentation of the form

$$0 \rightarrow \rho^*V' \otimes \mathcal{O}_\rho(-1) \rightarrow \rho^*W \rightarrow M' \rightarrow 0. \quad (2.21)$$

Since this construction is functorial, we obtain an injective morphism

$$\rho^*V \otimes \mathcal{O}_\rho(-1) \rightarrow \rho^*V' \otimes \mathcal{O}_\rho(-1), \quad (2.22)$$

thus an injective morphism $\rho^*V \rightarrow \rho^*V'$, and by adjunction, $V \subset V'$ in such a way that B extends. \square

There is a dual condition related to relative global generation.

Proposition 2.5. *Let $\rho : X \rightarrow C$ be a ruled surface, and suppose that M is a pure 1-dimensional sheaf on X . If M is ρ_* -acyclic, then M is relatively globally generated iff no quotient of M has the form $\mathcal{O}_f(-1)$ for some fiber f of ρ .*

Proof. If M is relatively globally generated, then

$$\mathrm{Hom}(M, \mathcal{O}_f(-1)) \subset \mathrm{Hom}(\rho^*\rho_*M, \mathcal{O}_f(-1)) \cong \mathrm{Hom}(\rho_*M, \rho_*\mathcal{O}_f(-1)) = 0. \quad (2.23)$$

For the converse, consider the natural map $\rho^*\rho_*M \rightarrow M$, viewed as a two-term complex. The terms in the complex are ρ_* -acyclic, and thus the derived direct image of the complex is

$$\rho_*\rho^*\rho_*M \cong \rho_*M, \quad (2.24)$$

so is exact. On the other hand, there is a spectral sequence converging to this result in which we first take the cohomology of the complex before taking higher direct images. Since ρ has 1-dimensional fibers, this spectral sequence stabilizes at the E_2 page, and we thus conclude that the cohomology sheaves of the complex have trivial direct image and higher direct image.

We thus conclude that if M is not globally generated, then M has a surjective morphism to a nonzero sheaf M' with $\rho_*M' = R^1\rho_*M' = 0$, so

$$M' \cong \rho^*\rho_*(M' \otimes \mathcal{O}_\rho(1)) \otimes \mathcal{O}_\rho(-1) \quad (2.25)$$

Now, $\rho_*(M' \otimes \mathcal{O}_\rho(1))$ cannot be 0, since that would force $M' = 0$. It thus admits a surjective map to some \mathcal{O}_p , which induces a surjective map from M' to a sheaf of the form $\mathcal{O}_f(-1)$. \square

Remark. In fact, it follows from this that if M is pure 1-dimensional and ρ_* -acyclic, then it is relatively globally generated iff $M \otimes \mathcal{O}_\rho(-1)$ is ρ_* -acyclic. Indeed, a surjection $M \rightarrow \mathcal{O}_f(-1)$ induces a surjection

$$R^1\rho_*(M \otimes \mathcal{O}_\rho(-1)) \rightarrow R^1\mathcal{O}_f(-2) \cong \mathcal{O}_{\pi(f)} \quad (2.26)$$

making the former sheaf nontrivial.

Similar conditions apply to ρ_* -acyclicity and torsion-freeness of ρ_*M .

Proposition 2.6. *Let $\rho : X \rightarrow C$ be a ruled surface, and let M be a pure 1-dimensional sheaf on X . Then ρ_*M is torsion-free iff $\text{Hom}(\mathcal{O}_f, M) = 0$ for all fibers f of ρ , and M is ρ_* -acyclic iff $\text{Hom}(M, \mathcal{O}_f(-2)) = 0$ for all f .*

Proof. For the first condition, we have

$$\text{Hom}(\mathcal{O}_f, M) \cong \text{Hom}(\rho^*\mathcal{O}_{\pi(f)}, M) \cong \text{Hom}(\mathcal{O}_{\pi(f)}, \rho_*M) \quad (2.27)$$

Since ρ_*M is torsion-free iff it has no maps from point sheaves, the first claim follows.

For the second, the same spectral sequence argument based on the complex $\rho^*\rho_*M \rightarrow M$ tells us that if M' is the cokernel of this natural map, then $R^1\rho_*M \cong R^1\rho_*M'$ and $\rho_*M' = 0$. Since M' is supported on finitely many fibers as before, it must have a quotient of the form $\mathcal{O}_f(-d)$ for some $d > 1$, and thus has a nontrivial morphism to $\mathcal{O}_f(-2)$. \square

If $\text{Hom}(\mathcal{O}_f, M) = 0$ but $\text{Hom}(\mathcal{O}_f(-1), M) \neq 0$, then any such morphism is necessarily injective; similarly, if $\text{Hom}(M, \mathcal{O}_f(-2)) = 0$ but $\text{Hom}(M, \mathcal{O}_f(-1)) \neq 0$, then any such morphism is surjective. We thus arrive at the final moduli problem: Classify pure 1-dimensional sheaves M on X with specified restriction to C_α and such that $\text{Hom}(M, \mathcal{O}_f(-1)) = \text{Hom}(\mathcal{O}_f(-1), M) = 0$ for all fibers f of the ruling.

We are also interested in understanding when W is trivial, for which we have the following numerical condition in the rational case.

Lemma 2.7. *Let $\rho : X \rightarrow \mathbb{P}^1$ be a Hirzebruch surface, and let M be a sheaf on X with 1-dimensional support. Then the following are equivalent:*

- (a) $H^0(M) = H^1(M) = 0$
- (b) M is ρ_* -acyclic and $\rho_*M \cong \mathcal{O}_{\mathbb{P}^1}(-1)^n$ for some $n \geq 0$.

Proof. Since ρ has 1-dimensional fibers, $R^p\rho_*M = 0$ for $p > 1$; since the generic fiber of $\text{supp}(M)$ over \mathbb{P}^1 is 0-dimensional, $R^1\rho_*M$ has 0-dimensional support. The Leray-Serre spectral sequence

$$H^p(R^q\rho_*M) \implies H^{p+q}(M) \quad (2.28)$$

thus implies isomorphisms

$$H^0(M) \cong H^0(\rho_*M) \quad H^2(M) \cong H^1(R^1\rho_*M) = 0 \quad (2.29)$$

and a short exact sequence

$$0 \rightarrow H^1(\rho_*M) \rightarrow H^1(M) \rightarrow H^0(R^1\rho_*M) \rightarrow 0. \quad (2.30)$$

In particular, $H^0(M) = H^1(M) = 0$ iff both ρ_*M and $R^1\rho_*M$ have vanishing cohomology. In particular, both must be isomorphic to a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(-1)$, and since $R^1\rho_*M$ has 0-dimensional support, it must be 0. \square

Thus the only way a 1-dimensional sheaf with $H^*(M \otimes \rho^* \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ could fail to have a presentation of the standard form is if $\text{Hom}(M, \mathcal{O}_f(-1)) \neq 0$ for some fiber f . (Even if it fails this last condition, the image of $\rho^* \rho_* M \rightarrow M$ gives us a subsheaf with standard presentation, and thus a subquotient with standard presentation satisfying maximality.)

When we considered elliptic difference equations above, this was in fact a simplification: in fact, the difference equations that occur in the theory of elliptic special functions have *theta function* coefficients in general. Algebraically speaking, A is really a matrix with coefficients in $\mathcal{L}_0 \otimes_{\mathcal{O}_{C_\alpha}} k(C_\alpha)$, where \mathcal{L}_0 is a degree 0 line bundle equipped with an isomorphism

$$\eta^* \mathcal{L}_0 \cong \mathcal{L}_0^* \quad (2.31)$$

such that the composition

$$\mathcal{L}_0 = \eta^* \eta^* \mathcal{L}_0 \cong \eta^* \mathcal{L}_0^* \cong \mathcal{L}_0 \quad (2.32)$$

is the identity. Again, Hilbert's Theorem 90 allows us to factor \mathcal{L}_0 , though now the nonuniqueness is more significant. If \mathcal{L}_0 has the above form, then it can (since we are over an algebraically closed field) be factored as

$$\psi : \mathcal{L}_0 \cong \eta^* \mathcal{L} \otimes \mathcal{L}^*, \quad (2.33)$$

for some line bundle \mathcal{L} . This line bundle is nonunique in two respects: the obvious one is that it can be twisted by any power of the line bundle $\pi_\eta^* \mathcal{O}_{\mathbb{P}^1}(1)$, but even modulo this, there are 8 possibilities for \mathcal{L} , and for each such choice, 2 possibilities for the isomorphism ψ . Indeed, we can twist \mathcal{L} by the degree 1 bundle corresponding to any ramification point of η ; if we twist by them all, this is the same as twisting by $\pi_\eta^* \mathcal{O}_{\mathbb{P}^1}(2)$, except that the isomorphism is multiplied by -1 . (In characteristic 2, the issue of nonuniqueness is somewhat more complicated. In general, modulo twisting $\pi_\eta^* \mathcal{O}_{\mathbb{P}^1}(1)$, the factorizations form a torsor over an abelian group scheme with structure $\mu_2 \cdot \text{Pic}^0(C_\alpha)[2] \cdot \mathbb{Z}/2\mathbb{Z}$.)

This nonuniqueness is related to the question of singularities at the four ramification points: we can no longer canonically distinguish between the two possible local rank 1 equations, in order to decide which one should be viewed as regular. Since we want to specify the singularity structure, we should view the factorization of \mathcal{L}_0 as part of the specification (and indeed, in all of the motivating cases, there is a natural choice of factorization making the equation regular at the fixed points of η for generic parameters). With this in mind, we again obtain a canonical factorization, except now B has the form

$$B : \pi_\eta^* V \rightarrow \pi_\eta^* W \otimes \mathcal{L}. \quad (2.34)$$

Again, two uses of adjunction allow us to extend this to a morphism of vector bundles on the Hirzebruch surface $\mathbb{P}(\pi_{\eta*} \mathcal{L})$, and thus further to the cokernel of that morphism. The above considerations extend immediately to the case of general \mathcal{L} .

Note that $\mathbb{P}(\pi_{\eta*} \mathcal{L})$ is isomorphic to the Hirzebruch surface F_1 iff \mathcal{L} has odd degree, and is otherwise isomorphic to F_0 or F_2 , with the latter precisely when \mathcal{L} is a power of $\pi_\eta^* \mathcal{O}_{\mathbb{P}^1}(1)$.

We should also note that using the freedom to twist \mathcal{L} by powers of $\pi_\eta^* \mathcal{O}_{\mathbb{P}^1}(1)$, we can assume that \mathcal{L}^* is represented by an effective divisor disjoint from the ramification locus. The resulting isomorphism $\mathcal{L} \cong \mathcal{O}_{C_\alpha}(-D)$ allows us to translate the problem back to one on F_2 , but with additional “apparent” singularities along D . (These are points where the obstructions to having symmetric solutions can be gauged away, possibly at the expense of introducing apparent singularities along other orbits of the infinite dihedral group.)

Although the generic anticanonical curve on F_2 is a smooth genus 1 curve disjoint from s_{\min} , there is significant scope for degeneration. We can view F_2 as the minimal desingularization of a

weighted projective space, and in this way anticanonical curves correspond to equations of the form

$$p_0(x, w)y^2 + p_2(x, w)y + p_4(x, w) = 0, \quad (2.35)$$

with $p_d(x, w)$ homogeneous of degree d . (Here w, x, y are generators of degrees 1, 1, and 2 of a graded algebra, and F_2 is the minimal desingularization of $\text{Proj}(k[w, x, y])$.) The anticanonical curve is disjoint from s_{\min} iff $p_0 \neq 0$, and thus we should consider equations

$$y^2 + p_2(x, w)y + p_4(x, w) = 0. \quad (2.36)$$

On any such curve, we have an involution $y \mapsto -p_2(x, w) - y$ which exchanges the two points on any given fiber. Given any such curve, the translation between matrices B on C_α and matrices on F_2 is quite explicit in terms of coordinates: express B in terms of the coordinates, and use the equation of C_α to eliminate any term of degree ≥ 2 in y . The resulting matrix, every coefficient of which is linear in y and weighted homogeneous, can now be viewed as a matrix on F_2 , and is a canonical extension of B .

There are several degenerate cases to consider. In characteristic not 2, we can complete the square to make $p_2 = 0$, and the degenerate cases are classified by the multiplicities of the zeros of p_4 (with an additional case when $p_2 = p_4 = 0$). These cases all extend to characteristic 2; there is one extra case in characteristic 2 ($p_2 = 0$, p_4 is not a square) for which we do not have a natural difference/differential equation interpretation, so do not consider below. Note that over a perfect field, any such curve is equivalent to the curve $y^2 = xw^3$.

- 211: C_α is integral, with a single node. Then C_α is isomorphic to \mathbb{P}^1 with 0 and ∞ identified, and η acts on this \mathbb{P}^1 as $z \mapsto \beta/z$ for some β . Up to a change of coordinates on F_2 , C_α has the equation

$$y^2 - xwy + \beta w^4 = 0, \quad (2.37)$$

with $w(z) = 1$, $x(z) = z + \beta/z$, $y(z) = z$. If we choose an automorphism $\tau_q : z \mapsto qz$, then the above construction applies to relate sheaves on F_2 to symmetric q -difference equations

$$v(qz) = A(z)v(z) \quad (2.38)$$

such that $A \in \text{GL}_n(k(z))$ satisfies $A(\beta/z) = A(z)^{-1}$. Note that this is singular at the node unless $A(0) = A(\infty) = 1$.

- 31: C_α is integral, with a single cusp. Then C_α is identified with \mathbb{P}^1 such that the cusp maps to ∞ and $\eta(z) = \beta - z$ for some β . Up to a change of coordinates on F_2 , C_α has the equation

$$y^2 - \beta w^2 y + xw^3 = 0, \quad (2.39)$$

with $w(z) = 1$, $x(z) = z(\beta - z)$, $y(z) = z$. These correspond to symmetric ordinary difference equations:

$$v(z + q) = A(z)v(z) \quad (2.40)$$

with $A \in \text{GL}_n(k(z))$ such that $A(\beta - z)A(z) = 1$. The equation is singular at the cusp unless $A(z) = 1 + O(1/z^2)$ as $z \rightarrow \infty$. (The symmetry then implies $A(z) = 1 + O(1/z^3)$.)

- 22: C_α is a union of two smooth components (isomorphic to \mathbb{P}^1) meeting in two distinct points, and η swaps the components. In suitable coordinates, C_α has the equation

$$y^2 - xwy = 0, \quad (2.41)$$

with components $y = xw$, $y = 0$; we may view $z = x/w$ as a common coordinate on the two components. Then any morphism B on F_2 as above specifies a *pair* of morphisms

$$B_1, B_2 : V \rightarrow \mathcal{O}_{\mathbb{P}^1}^n, \quad (2.42)$$

agreeing at 0 and ∞ . The corresponding A matrix on C_α is really a pair of inverse matrices, but we may simply view it as a single matrix $B_2^{-t} B_1^t$ on one component of C_α . In this way, we obtain a q -difference equation on \mathbb{P}^1 without any symmetry condition, and B_1, B_2 separate the zeros and poles of the equation. Singularities at the two nodes of C_α arise when $A(0) \neq 1$ or $A(\infty) \neq 1$ respectively.

- 4: C_α is a union of two smooth components which are tangent at a single point, and η swaps the components; C_α has equation $y^2 = w^2 y$, up to changes of coordinates. Again B specifies a pair of morphisms, which now agree to second order at ∞ . This corresponds to ordinary difference equations without symmetry, which are singular at ∞ unless $A(z) = 1 + O(1/z^2)$ as $z \rightarrow \infty$.
- 0: C_α is nonreduced, with equation $y^2 = 0$ after a change of coordinates. In this case, the degree 2 morphism $C_\alpha \rightarrow \mathbb{P}^1$ is no longer generically étale, so is not the quotient by an involution. However, we can now identify B with a pair of maps

$$B_0 : V \rightarrow \mathcal{O}_{\mathbb{P}^1}^n, \quad B_\omega : V \rightarrow \omega_{\mathbb{P}^1}^n, \quad (2.43)$$

giving a canonical factorization of a meromorphic matrix taking values in $\omega_{\mathbb{P}^1}$. Since there is a canonical connection on $\mathcal{O}_{\mathbb{P}^1}$, we may use this to interpret the meromorphic matrix with values in 1-forms as a meromorphic connection. This, of course, is just the standard translation between differential equations and sheaves on F_2 arising in the usual theory of Hitchin systems.

The construction in the nonsymmetric difference equation cases extends to one for general elliptic difference equations.

Example 2.1. Consider a general elliptic difference equation $\tau_q^* v = Av$ with $A \in \mathrm{GL}_n(k(C))$ for some genus 1 curve C . There is a natural factorization

$$A = B_\infty^{-t} B_0^t \quad (2.44)$$

where $B_0, B_\infty : V \rightarrow \mathcal{O}_C^n$ and V is a maximal vector bundle supporting such a factorization. (The existence of a meromorphic factorization is trivial (take $B_\infty = 1$, $B_0 = A^t$), and implies the existence of a unique maximal V as above.) The singularity structure of A then corresponds in a natural way to the cokernels of B_0 and B_∞ (giving zeros and poles respectively). The pair (B_0, B_∞) extends immediately to a morphism of vector bundles on the ruled surface $E \times \mathbb{P}^1$: just take $zB_\infty + wB_0$, where (z, w) are homogeneous coordinates on \mathbb{P}^1 . We can then recover the pair as the restriction of this morphism to the anticanonical curve $zw = 0$, a union of two disjoint copies of C . This is essentially just the construction for the Sklyanin integrable system (see [23]), the only difference being that the construction in the literature twists by a line bundle in order to absorb all of the poles, making $B_\infty = 1$, but making the ruled surface more complicated. (This corresponds to performing a sequence of elementary transformations centered at the points where the sheaf M meets the component $w = 0$ of the anticanonical curve.) More generally, we should allow difference equations on vector bundles, i.e., meromorphic (and meromorphically invertible) maps $A : V \rightarrow \tau_q^* V$. If there is a holomorphic isomorphism $A_0 : V \cong \tau_q^* V$, then one can divide by

A_0 to again reduce to a sheaf on $E \times \mathbb{P}^1$. By Atiyah's classification of vector bundles on smooth genus 1 curves, we find that $V \cong \tau_q^* V$ whenever V is a sum of indecomposable bundles of degree 0 (and this is necessary if q has infinite order). This is again an open condition on V (for degree 0 bundles, it is equivalent to semistability), and the isomorphism A_0 can be at least partially globalized (i.e., it exists on some open cover of the open subset); thus, just as in the rational cases, we can identify large open subsets of the moduli spaces of sheaves and of difference equations. (The main distinction is that the identification is no longer canonical, since A_0 is only determined up to scalars.) Note also that V is semistable of degree 0 iff there exists some line bundle \mathcal{L} of degree 0 such that $H^0(V \otimes \mathcal{L}) = H^1(V \otimes \mathcal{L}) = 0$; this gives an analogue of Lemma 2.7 for the elliptic case.

Example 2.2. Similarly, the Hitchin system corresponds to the analogous factorization for a meromorphic morphism

$$A : k(C) \rightarrow \omega_C \otimes_{\mathcal{O}_C} k(C), \quad (2.45)$$

and gives a sheaf on the anticanonical surface $\mathbb{P}(\mathcal{O}_C \oplus \omega_C)$. As before, the construction in the literature essentially absorbs the poles of A into the structure of the anticanonical surface, but this is just a sequence of elementary transformations. Once again, the “true” moduli space (of meromorphic connections on vector bundles) and the moduli space of sheaves can be identified along large open subsets; in this case, the requirement is that the vector bundle V admit a holomorphic connection. This is no longer an open condition (it is equivalent to every indecomposable summand having degree a multiple of the characteristic [8]), but is implied by open conditions, e.g., that V is semistable of degree 0 or stable of degree a multiple of the characteristic. It is also implied by the open condition that $H^0(V \otimes \mathcal{L}) = H^1(V \otimes \mathcal{L}) = 0$ for some line bundle \mathcal{L} of degree $g - 1$, though this is no longer equivalent to semistability.

Specifying an anticanonical curve on a smooth projective surface is tantamount to specifying a Poisson structure; more precisely, the structure is determined up to a scalar multiple, which can be fixed by a suitable choice of nonzero holomorphic differential on the anticanonical curve. The above examples account, up to birational maps respecting the Poisson structure, for almost every Poisson surface which is not symplectic, since it was shown in [42] that every such surface (apart from an exotic family of examples in characteristic 2) has the form $\mathbb{P}(\mathcal{O}_C \oplus \omega_C)$ in such a way that the section corresponding to ω_C is disjoint from the anticanonical curve. That our surfaces have this structure is significant since it follows from [22, 42] that the moduli space of sheaves with specified restriction to the anticanonical curve is symplectic (and the closure inside the moduli space of stable sheaves is Poisson).

The only missing example (up to isomorphism) is the Poisson structure on the characteristic 2 surface $F_2(\overline{\mathbb{F}_2})$ corresponding to the anticanonical curve $y^2 = xw^3$. (The exotic surfaces in characteristic 2 do not have Poisson moduli spaces, so we may feel free to ignore them.) This is an irreducible cuspidal curve as in case 31 above, but the degree 2 map to \mathbb{P}^1 is not étale, making the interpretation as a symmetric difference equation problematical. (Of course ordinary difference equations in finite characteristic are already somewhat problematical, since the infinite cyclic group acts via a finite quotient.)

There are a few other moduli problems that translate to sheaves on anticanonical rational surfaces that we want to consider.

Example 2.3. Let C_α be a smooth genus 1 curve, and let \mathcal{L} be a line bundle on C_α with $\deg \mathcal{L} \geq 0$. Consider the problem of classifying matrices $B \in \text{Mat}_n(\Gamma(\mathcal{L}))$ such that $\det(B) \neq 0$, up to left- and right- multiplication by constant matrices. For any choice of hyperelliptic involution η , we can encode such matrices via sheaves on the Hirzebruch surface $\mathbb{P}(\pi_{\eta*} \mathcal{L})$, essentially as above. The only additional condition we impose is that the bundle V must also be trivial. In the case $\deg(\mathcal{L}) = 3$,

we can also interpret the matrix as one over $\Gamma(\mathcal{O}_{\mathbb{P}^2}(1))$, and on \mathbb{P}^2 , the matrix is the minimal resolution of its cokernel, as in [7]. Note that since in this case we want both bundles to be trivial up to twist, we need to impose another numerical condition, which is straightforward to determine from the formula for V in terms of M that we gave above.

Example 2.4. On the Hirzebruch surface $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$, which we can view as the smooth quadric $xz = yw$ in \mathbb{P}^3 , we can apply the previous construction to the degenerate anticanonical curve $xz = 0$ (and the induced line bundle of degree 4). This has four components (two fibers and two sections), forming a quadrangle. Any linear combination of the coordinates on \mathbb{P}^3 is determined by its values at the four points of intersection of the quadrangle, and thus to specify a linear matrix B , it is equivalent to specify a quadruple (B_0, B_1, B_2, B_3) of scalar matrices. If these matrices are invertible, then the restriction of $\text{coker}(B)$ to $xz = 0$ is determined by its restrictions to the four components, and thus by the conjugacy classes of the matrices

$$B_0^{-1}B_1, B_1^{-1}B_2, B_2^{-1}B_3, B_3^{-1}B_0. \quad (2.46)$$

In other words, the problem of classifying sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ with the appropriate kind of presentation and restriction to the quadrangle is equivalent to the problem of classifying quadruples in $\text{GL}_n(k)$ with specified conjugacy classes and product 1. This is the four-matrix case of the “multiplicative Deligne-Simpson problem”, [11].

Example 2.5. Of course, we can obtain the three-matrix version of the multiplicative Deligne-Simpson problem by insisting that $B_3 = B_0$, or in other words that the sheaf meet the relevant component in \mathcal{O}_p^n where p is the point representing 1. If we blow this point up and blow down both the fiber and the section containing it, we obtain a moduli problem on \mathbb{P}^2 concerning sheaves with specified restriction to the triangle $xyz = 0$.

There does not appear to be any way to translate more general multiplicative Deligne-Simpson problems into the Poisson surface framework. For the additive problem, the situation is nicer.

Example 2.6. The Hirzebruch surface F_d for $d \geq 1$ has a unique section s_{\min} of negative self-intersection ($s_{\min}^2 = -d$). Given any $d + 2$ distinct fibers f_0, \dots, f_{d+1} , the divisor $2s_{\min} + \sum_i f_i$ is anticanonical, so we may take it as our curve C_α . Given any $(d + 2)$ -tuple of matrices C_0, \dots, C_{d+1} with $\sum_i C_i = 0$, we have a natural corresponding matrix with coefficients in $\mathcal{L}(s_{\min} + df)$. Indeed, we can coordinatize F_d in terms of a weighted projective space with a generator y of degree d , and consider the matrix

$$B = y + C(x, w) \quad (2.47)$$

such that $C(x, w)$ is equal to C_i on f_i . Then the restriction of B to f_i is given by the conjugacy class of C_i . In this way, we obtain the $(d + 2)$ -matrix additive Deligne-Simpson problem (classifying $(d + 2)$ -tuples of matrices with specified conjugacy classes and sum 0).

Remark. This, of course, is closely related to the problem of classifying Fuchsian differential equations with specified singularity structure; if we blow up a point of each fiber f_i then blow down f_i , we can in this way eliminate all components of C_α but the strict transform of s_{\min} . If we choose the centers of the elementary transformations carefully, we can arrange to end up at F_2 , and case 0 above.

If we forget the symmetry of a symmetric elliptic difference equation, we obtain a subspace of the moduli space of all elliptic difference equations, which we can understand in the following way. On the surface $C \times \mathbb{P}^1$, consider the involution $\eta \times (z \mapsto z^{-1})$. This preserves our standard choice of anticanonical curve, and more precisely preserves the corresponding Poisson structure (in contrast to $\eta \times 1$, say, which *negates* the Poisson structure). It follows that this involution acts on

the corresponding moduli space, again preserving the Poisson structure, and thus the fixed locus inherits a Poisson structure (at least in characteristic $\neq 2$). There are some difficulties in studying symmetric equations from this perspective, however. One is that, as we have seen, the notion of singularity should really take into account the symmetry, but another is that when working with moduli spaces, a fixed point merely indicates a sheaf which is *isomorphic* to its image under the symmetry. Since a sheaf only determines a matrix up to a choice of basis, not every point of the fixed locus actually corresponds to a symmetric equation. (The situation is not too dire, though: symmetric equations form a component of the fixed locus.) Note that the quotient of $E \times \mathbb{P}^1$ by the above involution is still an elliptic surface (with constant j invariant, and with two I_0^* fibers in characteristic not 2), and thus must be blown down eight times to reach a Hirzebruch surface. (We can arrange to reach the usual F_2 constructed from (C, η) , in which case the map from the elliptic surface blows up each fixed point of η twice.) This reflects both the fact that the notion of singularity changes and the fact that the fixed points of the moduli space are the equations which are symmetric *up to isomorphism*.

Similar comments apply if we try to relate symmetric and nonsymmetric difference equations in the q -difference and ordinary difference cases. More generally, we could consider any Poisson involution on one of our Poisson Hirzebruch surfaces. We find that the most general Poisson involution (again, in characteristic not 2) is again at the elliptic level, and is simply given by translation by a 2-torsion point p of $\text{Pic}^0(C)$; any other Poisson involution on F_2 is a degeneration of this (on some degenerate curve). Given any symmetric elliptic difference equation on the isogenous curve $C/\langle p \rangle$, we can interpret it as an equation on C (typically with twice as many singularities), and the corresponding sheaf will be invariant under the Poisson involution. Since any Poisson involution degenerates this, it in particular follows that the embedding of symmetric equations in the moduli space of nonsymmetric equations is a degeneration of this “quadratic transformation”. (So called because at the bottom, differential level, that is precisely what it is: performing a quadratic change of variables in the differential equation.) Once again, the mismatch between the two notions of singularity and the fact that equations can be symmetric up to isomorphism without being symmetric is reflected in the fact that the quotient by the involution is a (singular) del Pezzo surface of degree 4 with an $A_1 A_1 A_3$ configuration of -2 -curves. (One of the -2 -curves comes from the original -2 -curve, while the other four come from fixed points of the involution, two of which are on the original -2 -curve.)

There may also be some interesting phenomena related to anti-Poisson involutions of rational surfaces (which can be identified by the fact that they are hyperelliptic when restricted to the anticanonical curve). Though these remain anti-Poisson on the moduli space, they can be combined with a natural duality operation on sheaves to again obtain a Poisson involution on the moduli space. One example of this is the adjoint operation $A \mapsto A^{-t}$, see Section 8 below.

3 Blowdown structures on rational surfaces

In [42], we gave a construction for lifting sheaves (of homological dimension ≤ 1 , so in particular sheaves of pure dimension 1) through birational morphisms. (In the case of the direct image of a line bundle on a smooth curve, this is the obvious lift to the strict transform, but the construction applies more generally.) Moreover, up to “pseudo-twist”, we can lift any sheaf transverse to the anticanonical curve to some blowup on which it is *disjoint* from the anticanonical curve. (We will see that pseudo-twists correspond to certain canonical gauge transformations of difference equations, so we do not lose much generality by assuming we have such a lift.) As a result, we find that we want to consider sheaves on more general rational surfaces.

Once we have lifted to a blowup of our Hirzebruch surface, we encounter a new phenomenon: rational surfaces can be blown down to Hirzebruch surfaces in multiple ways. Although this is true in a mild sense for ruled surfaces of higher genus, in those cases we find that any two blowdowns to geometrically ruled surfaces are related by a sequence of elementary transformations (corresponding to a very mild transformation of the difference/differential equation). In contrast, rational surfaces no longer have a canonical rational ruling, and as a result a given sheaf on a rational surface will tend to have multiple qualitatively different interpretations as difference equations. (For instance, we will see that sheaves can correspond to symmetric or nonsymmetric q -difference equations depending on the choice of blowdown.)

With this in mind, we want to understand the set of possible ways to blow a rational surface down to a Hirzebruch surface. It turns out to be useful to record slightly more information than just a birational morphism to a Hirzebruch surface, and we thus make the following definition.

Definition 1. Let X be a rational surface with $X \not\cong \mathbb{P}^2$. A (Hirzebruch) blowdown structure on X is a chain Γ of morphisms

$$X = X_m \rightarrow X_{m-1} \rightarrow \cdots \rightarrow X_0 \rightarrow \mathbb{P}^1, \quad (3.1)$$

such that for $1 \leq i \leq m$, the morphism $X_i \rightarrow X_{i-1}$ is the blowup in a single point of X_{i-1} , while the morphism from $X_0 \rightarrow \mathbb{P}^1$ is a geometric ruling. Two blowdown structures will be considered equivalent if they fit into a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & X_{m-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & \mathbb{P}^1 \\ \parallel & & \downarrow & & \cdots & & \downarrow & & \downarrow \\ X & \longrightarrow & X'_{m-1} & \longrightarrow & \cdots & \longrightarrow & X'_0 & \longrightarrow & \mathbb{P}^1 \end{array} \quad (3.2)$$

Remark. Related structures have been studied in the case that the rational surface can be blown down to \mathbb{P}^2 , [50]; one also considers the related notion of an exceptional configuration (essentially the analogue for \mathbb{P}^2 of the notion of numerical blowdown structure below) [33]. Our considerations below are somewhat more general (since not every surface blows down to \mathbb{P}^2), but of course closely related; for instance [33] already saw the appearance of the root system E_{m+1} .

Note that in addition to keeping track of a factorization of the birational morphism, we also keep track of the ruling at the end (but only up to PGL_2). Most of the time, of course, the latter provides no information, since most Hirzebruch surfaces have a unique geometric ruling (and thus have a unique blowdown structure). The lone exception is $\mathbb{P}^1 \times \mathbb{P}^1$, and we note that above whenever we obtained $\mathbb{P}^1 \times \mathbb{P}^1$ from a problem of twisted difference equations, this came with a choice of ruling.

One reason for including the above information in the blowdown structure is that it allows us to associate a basis of $\mathrm{Pic}(X)$ to any blowdown structure. For each monoidal transformation $X_i \rightarrow X_{i-1}$, we have an exceptional curve e_i on X_i , and the total transform of this curve gives us a divisor on X , which we also denote e_i . Since

$$\mathrm{Pic}(X) \cong \mathrm{Pic}(X_0) \oplus \bigoplus_i \mathbb{Z}e_i, \quad (3.3)$$

it remains only to give a basis of $\mathrm{Pic}(X_0) \cong \mathbb{Z}^2$. One basis element is obvious, namely the class f of the fibers of the ruling. For the other, there is also an obvious choice, namely the class s_{\min} of a section with minimal self-intersection. This turns out not to be the best choice for our purposes,

however, as it gives us a countable infinity of different intersection forms to consider. A slightly different basis greatly reduces the number of cases. Define a divisor class

$$s := s_{\min} + \lfloor -s_{\min}^2/2 \rfloor f. \quad (3.4)$$

Since $s_{\min} \cdot f = 1$, $f^2 = 0$, we find that $s \cdot f = 1$, and s^2 is either 0 or -1 , depending on whether s_{\min}^2 was even or odd. With this in mind, we call a blowdown structure even or odd depending on the parity of s_{\min}^2 . (Note that if X_0 comes from a line bundle on a hyperelliptic genus 1 curve as in the previous section, then this choice of basis element essentially corresponds to choosing the bundle to have degree 1 or 2, as then $\mathcal{L}(s)$ agrees with the relative $\mathcal{O}(1)$.)

The basis we obtain has one of two possible intersection forms, depending on parity. In the even case, we have

$$s^2 = 0, \quad s \cdot f = 1, \quad f^2 = 0, \quad s \cdot e_i = f \cdot e_i = 0, \quad e_i \cdot e_j = -\delta_{ij}, \quad (3.5)$$

while in the odd case, we have the same, except $s^2 = -1$. The expansion of the canonical class in the basis again only depends on parity:

$$K_X = \begin{cases} -2s - 2f + \sum_{1 \leq i \leq m} e_i & \Gamma \text{ even} \\ -2s - 3f + \sum_{1 \leq i \leq m} e_i & \Gamma \text{ odd.} \end{cases} \quad (3.6)$$

And of course in either case we find $K_X^2 = 8 - m$. When $X_0 = F_1$, we can blow it down to \mathbb{P}^2 , suggesting an alternate basis in the odd case: replace f by $h = s + f$, the class of a line in \mathbb{P}^2 . This gives an orthonormal basis for $\text{Pic}(X)$, with $-K_X = -3h + s + \sum_i e_i$, but makes the effective cone look rather strange when $X_0 = F_{2d+1}$ for $d > 0$.

Note that we can recover the blowdown structure from the corresponding basis for $\text{Pic}(X)$: blow down e_m , then the image of e_{m-1} , etc., and construct a map $X_0 \rightarrow \mathbb{P}^1$ using f . Of course, not every basis with the correct intersection form will correspond to a blowdown structure, but we will eventually give an algorithm for determining when a given basis (expressed in terms of some original blowdown structure) also corresponds to a blowdown structure. In any event, we will define a *numerical* blowdown structure to be a basis of $\text{Pic}(X)$ having the same intersection form as an even or odd blowdown structure on X .

The surface X_1 was obtained by blowing up a point of X_0 , and we find that the fiber containing that point becomes a pair of -1 curves on X_1 , of divisor classes $e_1, f - e_1$. We thus obtain an alternate blowdown structure on X_1 by blowing down $f - e_1$, producing X'_0 differing from X_0 by an elementary transformation. The basis elements f and e_i for $i \geq 2$ are unchanged by this transformation, but e_1 and s are transformed as follows.

$$(s', e'_1) = \begin{cases} (s - e_1, f - e_1) & \Gamma \text{ even} \\ (s + f - e_1, f - e_1) & \Gamma \text{ odd} \end{cases} \quad (3.7)$$

Note that this swaps the even and odd cases, and if we perform the transformation twice, we end up back at the original blowdown structure.

Another natural way to transform a blowdown structure is to rearrange blowups. If the morphism $X_{i+2} \rightarrow X_i$ blows up two distinct points of X_i , then we can perform the blowdown in the other order, thus swapping the basis elements e_i and e_{i+1} . Unlike the elementary transformation case, this operation is *not* always legal, as when $X_{i+2} \rightarrow X_{i+1}$ blows up a point of e_i , there is no longer any choice in how to reach X_i . However, when it applies, it has a particularly nice action on the basis: it is simply the reflection with respect to the intersection form in the divisor class

$e_i - e_{i+1}$. Similarly, when $X_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, we obtain another blowdown structure by changing to the other ruling. This swaps the basis elements s and f , and is the reflection in the divisor class $s - f$.

In this way, we obtain a collection of m reflections in the even case, $m - 1$ in the odd case; if we perform an elementary transformation, the two sets mostly overlap, but we obtain a total of $m + 1$ different reflections in this way (assuming $m \geq 2$). The corresponding vectors are linearly independent, and are given by

$$s - f, f - e_1 - e_2, e_1 - e_2, \dots, e_{m-1} - e_m \quad (3.8)$$

in the even case and

$$s - e_1, f - e_1 - e_2, e_1 - e_2, \dots, e_{m-1} - e_m \quad (3.9)$$

in the odd case. Note that each one of these vectors is orthogonal to K ; we can see this either by direct calculation or by noting that the expansion of K depends only on the parity of the blowdown structure, so had better be invariant under the above reflections.

Lemma 3.1. *Suppose $m \geq 2$. Then the above sets of vectors give a basis for the orthogonal complement of K in $\text{Pic}(X)$. With respect to the negative of the intersection form, they form the set of simple roots of a Coxeter group of type E_{m+1} .*

Proof. In either case, we have $m + 1$ vectors, while $\text{Pic}(X)$ has rank $m + 2$, and thus we obtain bases of the orthogonal complement over \mathbb{Q} . Since the bases are obviously saturated (they are essentially triangular with unit diagonal), the first claim follows.

That the vectors are simple roots for a Coxeter system follows from the fact that their inner products are nonpositive (i.e., the intersections are nonnegative). To identify the system, note that the $e_i - e_{i+1}$ roots are the simple roots of a Coxeter group of type A_{m-1} , adjoining $f - e_1 - e_2$ extends this to D_m , and adjoining $s - f$ or $s - e_1$ as appropriate extends one of the short legs of the D_m Dynkin diagram. \square

Remark. Note the small m cases

$$\begin{aligned} E_3 &= A_1 \times A_2 \\ E_4 &= A_4 \\ E_5 &= D_5 \\ E_9 &= \tilde{E}_8, \end{aligned}$$

with E_6, E_7, E_8 as expected. When $m = 1$, we have only the root $s - f$ or $s - e_1$ as appropriate, and when $m = 0$, we have only $s - f$ in the even case, and no roots in the odd case.

With this in mind, we refer to the given vectors as the simple roots for the (numerical) blowdown structure. The corresponding simple reflections clearly give an action of $W(E_{m+1})$ on the set of numerical blowdown structures.

Lemma 3.2. *Suppose Γ is a blowdown structure for the rational surface X , and let σ be a simple root for Γ , with corresponding reflection r_σ . If σ is ineffective, then $r_\sigma \Gamma$ is a blowdown structure.*

Proof. To be precise, we mean here that if the numerical blowdown structure Γ comes from a blowdown structure, then so does $r_\sigma \Gamma$, as long as the divisor class σ is ineffective.

Using elementary transformations as appropriate, we may reduce to the cases $\sigma = s - f$ and $\sigma = e_i - e_{i+1}$. If $s - f$ is ineffective on X , it is certainly ineffective on X_0 , but then X_0 must be $\mathbb{P}^1 \times \mathbb{P}^1$ (if $X_0 \cong F_{2d}$ for $d > 0$, then $s_{\min} = s - df$ is effective), and we have already seen that the

reflection gives a blowdown structure. Similarly, if $e_i - e_{i+1}$ is ineffective on X , it is ineffective on X_{i+1} , which implies that $X_{i+1} \rightarrow X_{i-1}$ blows up two distinct points of X_i , so that we need merely blow up the points in the opposite order. \square

Remark. The roots of the A_{m-1} subsystem act without changing the Hirzebruch surface X_0 , by permuting the distinct points being blown up. Similarly, the simple reflections of the D_m subsystem leave the rational ruling invariant. If we combine those reflections with the action of the elementary transformation, we obtain a group of type C_m acting on the different ways to blow down to a Hirzebruch surface compatibly with the given ruling.

Given a blowdown structure Γ , call an element $w \in W(E_{m+1})$ *ineffective* if there exists a word $w = r_1 r_2 \cdots r_l$ with each r_i a simple reflection such that the corresponding simple root is ineffective for the relevant blowdown structure, $r_{i+1} r_{i+2} \cdots r_l \Gamma$. (That this numerical blowdown structure comes from an actual blowdown structure follows by an easy induction.) In particular, if w is ineffective, then $w\Gamma$ is a blowdown structure.

We thus need to understand the effective simple roots. By a “ $-d$ -curve” on a rational surface, we mean a smooth rational curve of self-intersection $-d$.

Lemma 3.3. *Let X be a rational surface with blowdown structure Γ and $K_X^2 < 8$. Then any effective simple root σ can be decomposed as a nonnegative linear combination of $-d$ -curves with $d > 0$. There is at most one fixed component of σ not orthogonal to σ , and that component has self-intersection ≤ -2 , with equality only if σ is a -2 -curve.*

Proof. If $\sigma = e_i - e_{i+1}$ is effective, then it is the total transform of a -2 curve on X_{i+1} . At each later step in the blowing up process, either we blow up a point not on the total transform, in which case the decomposition is unchanged, or we blow up a point on the total transform, in which case we acquire an additional component e_j , and the component(s) containing the center of the monoidal transform have their self-intersection decreased by 1. Thus by induction every component is a $-d$ -curve for some $d > 0$. We also see that σ is uniquely effective, so every component is fixed, but only the strict transform of the original -2 curve is not orthogonal to σ . The case $\sigma = f - e_1 - e_2$ follows by elementary transformation.

For the remaining case, assume for convenience that Γ is even, so the remaining root is $s - f$. This is effective precisely when $X_0 \cong F_{2d}$ with $d > 0$, when we can write it in the form

$$s - f = s_{\min} + (d - 1)f. \quad (3.10)$$

This same decomposition applies to X_m , so that the only fixed components are those of the total transform of s_{\min} , to which the previous calculation applies. On the other hand, we could choose the fibers in this decomposition to all pass through the point blown up on X_1 , obtaining a decomposition

$$s - f = s_{\min} + (d - 1)(f - e_1) + (d - 1)e_1, \quad (3.11)$$

or

$$s - f = (s_{\min} - e_1) + (d - 1)(f - e_1) + de_1 \quad (3.12)$$

on X_1 , the latter when the point being blown up is on s_{\min} . The components of this decomposition are all $-e$ -curves for varying $e > 0$, and as before this property is preserved on taking the total transform to $X_m = X$. \square

Since reflections in ineffective simple roots take blowdown structures to blowdown structures, we can define a groupoid (the *strict* groupoid of blowdown structures on X) as follows: the objects are the blowdown structures on X , while the morphisms are given by the actions of ineffective elements of $W(E_{m+1})$.

Theorem 3.4. *If X is a rational surface with $K_X^2 < 8$, then the strict groupoid of blowdown structures on X has precisely two isomorphism classes, one for each parity of blowdown structure. (If $K_X^2 = 8$, the groupoid has only one isomorphism class.)*

In other words, any two blowdown structures on X with the same parity are related by a sequence of reflections in ineffective simple roots. For $K_X^2 < 8$, elementary transformations imply that we need only consider the even parity case.

Proof. If $K_X^2 = 8$, this is obvious, as either there is only one blowdown structure or $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, and there are two blowdown structures related by reflection in $s - f$. Now, suppose $K_X^2 = 7$, and fix an even blowdown structure on X . To blow X down to a Hirzebruch surface, we must blow down a -1 curve, which in particular gives a divisor class D such that $D^2 = D \cdot K_X = -1$. There are only three such divisor classes, namely $D \in \{e_1, s - e_1, f - e_1\}$. Only the case $D = e_1$ could blow down to an even Hirzebruch surface, since the other two cases have classes of odd self-intersection in their orthogonal complements. In other words, a surface with $K_X^2 = 7$ blows down to a unique even Hirzebruch surface, and thus the even blowdown structures on X are bijective with the even blowdown structures on this Hirzebruch surface. (Note also that $f - e_1$ is always a -1 curve, while $s - e_1$, related to it by a simple reflection, is either a -1 curve or decomposes as $s - e_1 = (s - f) + (f - e_1)$, depending on whether $s - f$ is effective.)

For $K_X^2 < 7$, we may induct on m , and thus reduce to the question of showing that every -1 curve on X can be moved to e_m by a sequence of reflections in ineffective simple roots. Again, we may assume we are starting from an even blowdown structure, conjugating by elementary transformations as appropriate. Let

$$E = ns + df - \sum_i r_i e_i \quad (3.13)$$

be the class of the given -1 curve. Note that $n \geq 0$ since f is nef (it always has a representative which is smooth of self-intersection 0).

If $E \cdot \sigma < 0$ for some simple root, then the root must be ineffective, since otherwise E would be a fixed component of self-intersection -1 not orthogonal to the simple root. In particular, we can always perform the corresponding reflection, and this makes the vector $(n, d, -r_1, \dots, -r_{m-1})$ lexicographically smaller. Since the reflections preserving n form a finite group (of type D_m), we conclude that after finitely many reflections in ineffective simple roots, we will obtain a divisor such that $E \cdot \sigma \geq 0$ for every simple root σ . We may also assume $E \cdot e_m \geq 0$, since otherwise $E = e_m$.

For such a divisor, we have the inequalities

$$d \geq n \geq r_1 + r_2; \quad r_1 \geq r_2 \geq \dots \geq r_m \geq 0. \quad (3.14)$$

Since $(E - e_m) \cdot K_X = 0$, we may also express $E - e_m$ as a linear combination of simple roots:

$$E - e_m = n(s - f) + (n + d)(f - e_1 - e_2) + (n + d - r_1)(e_1 - e_2) + \sum_{2 \leq k \leq m-1} (1 + \sum_{k < i} r_i)(e_i - e_{i+1}), \quad (3.15)$$

clearly a nonnegative linear combination. Since

$$E \cdot (E - e_m) = -1 - r_m < 0, \quad (3.16)$$

we obtain a contradiction. \square

Remark 1. We can adapt this to an algorithm for testing whether a given class is a -1 -curve (and thus whether a given numerical blowdown structure comes from an actual blowdown structure): reflect in simple roots with $E \cdot \sigma < 0$ until one of the roots is effective, $E \cdot f < 0$, or $E = e_m$. Then E is a -1 -curve iff the last termination condition holds.

Remark 2. Of course, we could obtain a groupoid with a single isomorphism class by including morphisms of the form $w\epsilon$ where ϵ is the elementary transformation, but this is somewhat inconvenient, since the morphisms no longer correspond directly to elements of a group.

Any -2 -curve on X is a (real) root of the root system E_{m+1} . More precisely, we have the following.

Proposition 3.5. *Suppose D is the class of a -2 curve. Then there exists a blowdown structure on X for which D is a simple root.*

Proof. Let Γ be an even blowdown structure on X , and write

$$D = ns + df - \sum_{1 \leq i \leq m} r_i e_i \quad (3.17)$$

as before. Since $D \cdot K_X = 0$, we can expand D as a linear combination of simple roots

$$D = n(s - f) + (n + d)(f - e_1 - e_2) + (n + d - r_1)(e_1 - e_2) - \sum_{2 \leq k \leq m-1} \left(\sum_{k < i} r_i \right) (e_i - e_{i+1}), \quad (3.18)$$

and thus find as before that $D \cdot \sigma < 0$ for some simple root σ . Once more, D would have to be a fixed component of σ if σ were effective, and thus either $D = \sigma$ or σ is ineffective. As before, in the latter case, reflecting makes D lexicographically smaller, so this process must terminate. \square

Remark. Again, this translates to an algorithm for testing whether a given divisor class is represented by a -2 -curve, which is formally very similar to the algorithm of [27] for testing whether a local system is rigid.

In the case of a surface with a chosen anticanonical curve, there is a related groupoid with more morphisms and nontrivial stabilizers. Call a simple reflection $s \in S(E_{m+1})$ *admissible* for the blowdown structure Γ (and the anticanonical curve C_α) if the corresponding simple root is either ineffective or has intersection 0 with every component of C_α . Although $s\Gamma$ is no longer a blowdown structure when s is effective but admissible, we define a modified action as follows. If s is ineffective, then $s \cdot \Gamma := s\Gamma$, while if s is effective but admissible, then $s \cdot \Gamma := \Gamma$.

More generally, call an element $w \in W(E_{m+1})$ *admissible* for Γ if there exists a word

$$w = s_1 s_2 \cdots s_l \quad (3.19)$$

for w such that for each $1 \leq i \leq l$, s_i is admissible for

$$s_{i+1} \cdot s_{i+2} \cdots s_{l-1} \cdot s_l \cdot \Gamma. \quad (3.20)$$

In this case, we also call the given word *admissible*.

Proposition 3.6. *If w is admissible for Γ , then every reduced word for w is admissible for Γ . Moreover, if*

$$w = s_1 \cdots s_l = s'_1 \cdots s'_{l'} \quad (3.21)$$

are two admissible words representing w , then

$$s_1 \cdot s_2 \cdots s_{l-1} \cdot s_l \cdot \Gamma = s'_1 \cdot s'_2 \cdots s'_{l'-1} \cdot s'_{l'} \cdot \Gamma. \quad (3.22)$$

Proof. We first note that if s is admissible for Γ , then it is also admissible for $s \cdot \Gamma$; either s is ineffective and remains so, or s is a -2 curve, and $s \cdot \Gamma = \Gamma$. Either way, we find $s \cdot s \cdot \Gamma = \Gamma$. In other words, if a reflection occurs twice in a row in an admissible word, we can remove the pair without affecting admissibility or the final blowdown structure.

Since any word can be transformed into a reduced word by a sequence of braid relations and removal of repeated reflections, and any two reduced words are related by a sequence of braid relations, it remains only to show that the claim holds for braid relations. In other words, given a braid relation in $W(E_{m+1})$, we need to show that either both sides are inadmissible or both sides are admissible and produce the same blowdown structure.

Let s, t be simple reflections. If both are inadmissible, there is nothing to prove, so suppose that s is admissible for Γ . Then we observe that if t is inadmissible for Γ , then it is also inadmissible for $s \cdot \Gamma$. Thus only the case that s and t are both admissible need be considered. If the braid relation is $st = ts$, then we need merely check that the relation holds in each of the four cases (s effective or not, t effective or not).

Thus suppose the braid relation is $sts = tst$. Let r_s, r_t be the corresponding simple roots. If $r_s + r_t$ is ineffective, then either r_s, r_t are both ineffective (and the braid relation follows from the fact that the action agrees with the linear action) or precisely one (say r_t) is effective. But then we find that r_s is effective in the blowdown structure $t \cdot s \cdot \Gamma$, and thus

$$s \cdot t \cdot s \cdot \Gamma = t \cdot s \cdot t \cdot \Gamma = t \cdot s \cdot \Gamma. \quad (3.23)$$

If $r_s + r_t$ is effective and r_t is effective, then r_s is also effective. Indeed, $(r_s + r_t) \cdot r_t = -1$, and thus any representative of $r_s + r_t$ contains r_t as a component, implying $r_s + r_t - r_t$ effective. Thus in this case, we have

$$s \cdot t \cdot s \cdot \Gamma = t \cdot s \cdot t \cdot \Gamma = \Gamma, \quad (3.24)$$

since the blowdown structure never changes.

Finally, we have the case $r_s + r_t$ effective but r_s, r_t are ineffective. Relative to the blowdown structure $t \cdot \Gamma$, r_s is effective, and thus

$$s \cdot t \cdot s \cdot t \cdot \Gamma = t \cdot s \cdot t \cdot t \cdot \Gamma = t \cdot s \cdot \Gamma \quad (3.25)$$

(or both sides are undefined, if r_s is inadmissible for $t \cdot \Gamma$). If both sides are defined, then s is admissible for both blowdown structures, and thus

$$t \cdot s \cdot t \cdot \Gamma = s \cdot s \cdot t \cdot s \cdot t \cdot \Gamma = s \cdot t \cdot s \cdot t \cdot t \cdot \Gamma = s \cdot t \cdot s \cdot \Gamma, \quad (3.26)$$

and we are done. \square

If we use admissible elements in place of effective elements in defining the groupoid of blowdown structures, the resulting “weak” groupoid has nontrivial stabilizers, a conjugacy class of reflection subgroups of $W(E_{m+1})$.

Proposition 3.7. *The stabilizer of Γ in the weak groupoid of blowdown structures is the reflection subgroup of $W(E_{m+1})$ generated by reflections in -2 -curves disjoint from the anticanonical curve.*

Proof. Given a -2 -curve v disjoint from C_α , let w be an effective element of $W(E_{m+1})$ such that v is a simple root σ in $w\Gamma$. Then r_σ stabilizes $w\Gamma$, so $r_v = w^{-1}r_\sigma w$ stabilizes Γ .

Conversely, consider an admissible reduced word w stabilizing Γ . If every reflection in w is ineffective, then w acts linearly, and since it stabilizes a basis, we have $w = 1$. Otherwise, we can write

$$w = w_1 r w_2 \quad (3.27)$$

where r is an effective but admissible simple reflection, w_2 is ineffective, and $\ell(w) = \ell(w_1) + \ell(w_2) + 1$. Since r is effective, it stabilizes $w_2 \cdot \Gamma = w_2\Gamma$. We thus conclude that we can factor

$$w = w_1 w_2 (w_2^{-1} r w_2) \quad (3.28)$$

where both $w_1 w_2$ and $w_2^{-1} r w_2$ are admissible elements stabilizing Γ . The second factor is a reflection in the -2 -curve corresponding to r in $w_2\Gamma$ (which is admissible, so disjoint from C_α), while the first factor has length strictly smaller than $\ell(w)$. Thus by induction, w can be written as a product of reflections in -2 -curves disjoint from C_α . \square

Note from [20] that the -2 -curves disjoint from C_α can be determined in the following way: restriction to C_α gives a natural homomorphism $\text{Pic}(X) \rightarrow \text{Pic}(C_\alpha)$, and the -2 -curves are precisely the simple roots in the system of positive roots in the kernel of this homomorphism. This is easy to see from our perspective, as it reduces to checking when a simple root of E_{m+1} is a -2 -curve disjoint from C_α .

Given an anticanonical rational surface X , there is a natural combinatorial invariant of blowdown structures, namely how the components of C_α (which we fix an ordering of) are expressed in terms of the corresponding basis. That is, if we fix an ordered decomposition

$$C_\alpha = \sum_i c_i C_i \quad (3.29)$$

where the C_i are the distinct components of C_α (so each $c_i > 0$), then given any blowdown structure, we may associate the sequence of pairs (c_i, v_i) where $v_i \in \mathbb{Z}^{m+2}$ is the image of $C_i \in \text{Pic}(X)$ under the isomorphism $\text{Pic}(X) \cong \mathbb{Z}^{m+2}$ corresponding to Γ . (If C_α is integral, this invariant simply distinguishes between even and odd blowdown structures.) If we fix X and a decomposition of C_α , this invariant takes on only finitely many values as we vary Γ . In fact, something much stronger holds: if we take the union over all anticanonical surfaces with a chosen decomposition C_α , then there are only finitely many possibilities for any given value of $\min_i(v_i^2)$.

Indeed, if we put a lower bound on the self-intersections of the components of C_α , then this implies a lower bound on the self-intersections of any $-d$ -curve on X (since any $-d$ -curve with $d > 2$ has negative intersection with C_α , so is a component). In particular, this gives only finitely many possible Hirzebruch surfaces that X can be blown down to. On a given Hirzebruch surface, there are only finitely many combinatorially distinct decompositions of anticanonical curves, and as we blow up points, the change in invariant only depends on the set of components containing the point being blown up.

In particular, this combinatorial type splits the weak groupoid of blowdown structures into finitely many groupoids. We observe that each of these groupoids is a quotient groupoid G/H for some $G \subset W(E_{m+1})$, where H is the group generated by reflections in -2 -curves disjoint from C_α . Indeed, whether a simple root is admissible only depends on the combinatorial type, and thus the admissible elements of $W(E_{m+1})$ preserving the combinatorial type form a group. This group is certainly contained in the stabilizer of the sequence of vectors corresponding to the components of C_α , and itself contains a reflection group.

Proposition 3.8. *Suppose ρ is a positive root which is orthogonal to every component of C_α . Then the corresponding reflection is admissible.*

Proof. It suffices to consider the case that ρ is *simple* among the root system of positive roots orthogonal to every component of C_α . Then we claim that there is a blowdown structure in which ρ is a simple root of E_{m+1} . If ρ is already simple, this is immediate. Otherwise, let σ be a simple

root such that $\sigma \cdot \rho < 0$. If σ is ineffective, we may reflect in σ and proceed by induction. Otherwise, σ is effective, and some component c of σ which is not a component of C_α satisfies $c \cdot \rho < 0$. Since $f \cdot \rho \geq 0$ for every positive root, we have $c \neq f$, and thus by the proof of Lemma 3.3, c must be a fixed component of σ . If $c^2 = -2$, then it is orthogonal to every component of C_α , but then the fact that $c \cdot \rho < 0$ contradicts simplicity of ρ unless $\rho = c$, in which case reducing to a simple root of E_{m+1} is straightforward.

Otherwise, by the classification of fixed components of -2 -curves, we find that $c = e_i$ for some i , and that $e_j \cdot \sigma = 0$ for $j \geq i$. Since the only positive roots satisfying $\rho \cdot e_i < 0$ are those of the form $e_i - e_j$ for some $j > i$, we conclude that $\rho \cdot \sigma = (e_i - e_j) \cdot \sigma = 0$, a contradiction. \square

Remark. In general, the full stabilizer need not be a reflection subgroup. For instance, let $X = X_8$ be a rational elliptic surface with an anticanonical curve of Kodaira type I_3^* (corresponding to the root system \tilde{D}_7). Since a subsystem of type D_7 in E_8 has trivial stabilizer, we conclude that the stabilizer must be contained in the translation subgroup of $W(E_9) = W(\tilde{E}_8)$ in this case. Any translation will add some multiple of $-K_X$ to the different components, preserving the property that the relevant linear combination is $-K_X$; it follows that the stabilizer contains a corank 7 subgroup of the translation subgroup of E_8 . In other words, the stabilizer is isomorphic to \mathbb{Z} , so is certainly not a reflection subgroup! (We can also directly verify in this case that the generator of the stabilizer is admissible.) The stabilizer also fails to be a reflection subgroup of $W(E_9)$ when the anticanonical curve has Kodaira type I_7 , and in one of the two ways it can have Kodaira type I_8 . In the I_8 case it is again isomorphic to \mathbb{Z} , while in the I_7 case it is isomorphic to $W(\tilde{A}_1) \times \mathbb{Z}$.

Corollary 3.9. *If X is a rational surface with an integral anticanonical curve, then the weak groupoid of blowdown structures on X is a union of two isomorphic quotient groupoids of the form $W(E_{m+1})/H$ where H is the group generated by reflections in the -2 curves of X .*

Remark. The reader should be cautioned that a reflection subgroup of an infinite Coxeter group need not have finite rank. Indeed, an example was given in [18, Ex. 2.8] of a rational surface with (nodal) integral anticanonical curve and infinitely many -2 -curves, and thus the stabilizers in the corresponding groupoid have infinite rank.

4 Divisors on rational surfaces

4.1 Nef divisors

Given a rational surface and blowdown structure, one natural question which arises is whether a given vector corresponds to an effective divisor class, or one with an integral representative. For the latter, it will be helpful to also have an answer to the question of which vectors correspond to nef divisor classes. This is complicated in general, but in the anticanonical case, is quite tractable. Note that as with the above algorithms for recognizing -2 - and -1 -curves, the algorithms below only depend on (a) the decomposition of C_α in some initial choice of blowdown structure, and (b) the kernel of the natural homomorphism $\text{Pic}(X) \rightarrow \text{Pic}(C_\alpha)$. (The latter is not particularly tractable to compute in general, but in fact all we really need is the ability to test membership in the kernel.)

One answer to this question was given in [32]: the monoid of effective divisors (assuming $K_X^2 < 8$) is generated by the $-d$ -curves with $d > 0$ and $-K_X$. In principle, this gives a way of testing whether a vector is nef: simply check that it has nonnegative intersection with $-K_X$ and every $-d$ -curve. Of course, this is not an actual algorithm, for the simple reason that a rational

surface can have infinitely many smooth curves of negative self-intersection. (In fact, this is the *typical* behavior!)

We do, however, obtain the following. Given an anticanonical rational surface X and a blowdown structure Γ for X , define the *fundamental chamber* to be the monoid in $\text{Pic}(X)$ consisting of classes having nonnegative intersection with every simple root for Γ . Note also that there are only finitely many $-d$ -curves on X with $d > 2$, since any such curve must be a component of $-K_X$.

Proposition 4.1. *Suppose X is an anticanonical rational surface with $K_X^2 \leq 6$, and let Γ be a blowdown structure on X . Let D be a divisor class in the fundamental chamber of Γ . Then D is nef iff $D \cdot C_\alpha \geq 0$, $D \cdot e_m \geq 0$, and D has nonnegative self-intersection with every $-d$ -curve with $d > 2$. If $C_\alpha^2 \geq 0$, then we can omit the condition $D \cdot C_\alpha \geq 0$.*

Proof. That nef divisors satisfy the given conditions is obvious, so it remains to show that the given conditions imply that D is nef. Since we have assumed $D \cdot C_\alpha \geq 0$, it remains to verify that it has nonnegative intersection with every $-d$ -curve for $d > 0$. We have also assumed this for $d > 2$, so only the cases of -2 - and -1 -curves remain. Any -2 -curve is a positive root of E_{m+1} , so is a nonnegative linear combination of simple roots. By assumption, D has nonnegative intersection with every simple root, so nonnegative intersection with every positive root. Similarly, we saw above that any -1 -curve can be written as e_m plus a nonnegative linear combination of simple roots, so again D has nonnegative intersection with every -1 -curve.

If $K_X^2 \geq 0$, then we can write K_X as a nonnegative linear combination of simple roots and e_m , so can omit the corresponding condition. \square

Remark. Something similar holds when $K_X^2 = 7$, except that we must also assume $D \cdot (f - e_1) \geq 0$. In any event, we can readily write down the effective and nef monoids when $K_X^2 = 7$. Indeed, if $X_0 \cong F_{2d}$ and $X_1 \rightarrow X_0$ blows up a point of s_{\min} (which we arrange to occur if $d = 0$), we have

$$\begin{aligned} \text{Eff}(X_1) &= \langle s - df - e_1, f - e_1, e_1 \rangle \\ \text{Nef}(X_1) &= \langle f, s + df, s + (d+1)f - e_1 \rangle, \end{aligned}$$

while if $d > 0$ and $X_1 \rightarrow X_0$ blows a point not on s_{\min} , we have

$$\begin{aligned} \text{Eff}(X_1) &= \langle s - df, f - e_1, e_1 \rangle \\ \text{Nef}(X_1) &= \langle f, s + df, s + df - e_1 \rangle. \end{aligned}$$

Since $s - df = s_{\min}$, it is clear that the putative generators for $\text{Eff}(X_1)$ are effective. Similarly, we find that the putative generators for $\text{Nef}(X_1)$ have nonnegative self-intersection, and can be represented by integral divisors, so are nef. Since in each case the two bases are dual to each other, they must actually be the effective and nef monoids. (The corresponding bases for the monoids relative to an odd blowdown structure can of course be obtained by an elementary transformation.) Note that in either case, $\text{Eff}(X_1)$ is a simplicial cone generated by $-e$ -curves with $e < 0$.

Since nef divisors are effective ([20, Cor. II.3]), we also conclude that any class D satisfying the above hypotheses is effective. In fact, we can do better: we can give an explicit effective divisor representing D .

Proposition 4.2. *With hypotheses as above, D can be written as a nonnegative linear combination of $-d$ -curves and $-K_X$.*

Proof. Suppose first that $D \cdot e_m = 0$, so that D is the total transform of a divisor on X_{m-1} . If $m > 2$, then this divisor on X_{m-1} is itself in the fundamental chamber, and has nonnegative intersection

with $-K_{X_{m-1}}$. We can thus decompose it into $-d$ -curves and copies of the anticanonical curve on X_{m-1} . The total transform of a $-d$ -curve is either a $-d$ -curve or the sum of a $-(d+1)$ -curve and e_m , while the total transform of the anticanonical curve on X_{m-1} is $C_\alpha + e_m$, and thus the decomposition on X_{m-1} induces a decomposition on X_m which again has the desired form.

Similarly, if $m = 2$ and $D \cdot e_2 = 0$, then we still find that D is the total transform of a nef divisor on X_1 , and thus obtain the desired decomposition of D by expanding it in the basis of the simplicial cone $\text{Eff}(X_1)$.

Finally, suppose $D \cdot e_m > 0$. Then we claim that $D - C_\alpha$ satisfies the original hypotheses. Indeed, if C is a $-d$ -curve for $d > 2$, then

$$(D - C_\alpha) \cdot C = D \cdot C + (d - 2) > D \cdot C, \quad (4.1)$$

while if σ is a simple root, then $(D - C_\alpha) \cdot \sigma = D \cdot \sigma \geq 0$. In addition, $(D - C_\alpha) \cdot e_m = D \cdot e_m - 1 \geq 0$. Finally, we have

$$(D - C_\alpha) \cdot C_\alpha = D \cdot C_\alpha - C_\alpha^2. \quad (4.2)$$

Either $C_\alpha^2 < 0$, so the inequality becomes stronger, or $C_\alpha^2 \geq 0$, and the inequality is redundant. Either way, $D - C_\alpha$ satisfies all of the hypotheses, and we obtain an explicit decomposition of the given form. \square

Lemma 4.3. *If D is a nef divisor on the rational surface X , then there exists a blowdown structure (of either parity) such that D is in the fundamental chamber. Moreover, the representation of D in the basis corresponding to such a blowdown structure depends only on the parity. In addition, if e is a -1 -curve with $e \cdot D = 0$, then the blowdown structure can be chosen in such a way that $e_m = e$.*

Proof. Choose a blowdown structure of the desired parity on X . If D is not already in the fundamental chamber, then there exists a simple root σ such that $D \cdot \sigma < 0$. Since D is nef, σ cannot be effective, and thus we can apply the corresponding reflection. Either σ is in the subsystem of type D_m (which can only occur finitely many times in a row, since that subgroup is finite), or it decreases $D \cdot f$. The latter is nonnegative since f is effective, and thus the process will terminate after a finite number of steps.

For uniqueness, suppose D is in the fundamental chamber of both Γ and Γ' , two blowdown structures of the same parity. Then there exists an ineffective element $w \in W(E_{m+1})$ such that $\Gamma' = w\Gamma$. If $w = 1$, then we are done; otherwise, there is an ineffective simple root σ of Γ such that $w\sigma$ is negative (the last root in some reduced word for w). Since D is in the fundamental chamber for both Γ and Γ' , it has nonnegative intersection with every positive root of either blowdown structure. Thus $D \cdot \sigma \geq 0$ since σ is positive for Γ , and $D \cdot (-\sigma) \geq 0$ since $-\sigma$ is positive for Γ' . In other words, $D \cdot \sigma = 0$, and thus the reflection in σ does not change the expansion of D in the standard basis. The claim follows by induction on the length of the reduced word for w .

Finally, if Γ is any blowdown structure with $e_m = e$, then $D \cdot (e_{m-1} - e_m) \geq 0$, and thus the algorithm for putting D in the fundamental chamber will never try to apply the corresponding reflection, so will never change e_m . \square

Remark. Uniqueness is of course a standard fact from Coxeter theory when we restrict to D in the root lattice, and the above argument is adapted from the standard one.

This then gives us the desired algorithm for testing whether a divisor is nef: First check that it has nonnegative intersection with every $-d$ -curve with $d > 2$ and with C_α , then repeatedly attempt to reflect in simple roots with $D \cdot \sigma < 0$. If at any step we have σ effective, $D \cdot f < 0$ or $D \cdot e_m < 0$, then D is not nef; otherwise, we terminate in the fundamental chamber, and conclude that D is nef.

4.2 Effective divisors

A similar algorithm works for testing whether a divisor D is effective. We assume $K_X^2 < 7$, since otherwise the effective cone is simplicial, so testing whether D is effective is just linear algebra.

Again, we start by choosing any blowdown structure for X , and if at any step in the process we obtain a divisor with $D \cdot f < 0$, we halt with the conclusion that our divisor was ineffective. We perform the following steps, as specified.

1. If there exists a component C of C_α such that $C^2, D \cdot C < 0$, then replace D by $D - C$, and repeat step 1.
2. If $D \cdot e_m < 0$, then replace D by $D + (D \cdot e_m)e_m$ and go back to step 1.
3. If D is in the fundamental chamber, conclude that the original divisor was effective. Otherwise, choose the lexicographically smallest simple root such that $D \cdot \sigma < 0$. If σ is effective, replace D by $D - \sigma$ and go back to step 1; otherwise, replace Γ by $r_\sigma \Gamma$ and go back to step 2.

To see that this algorithm works, we note as before that f is nef, so any divisor with $D \cdot f$ is not effective. Whenever we replace D by $D - C$ in step 1, C is an integral curve of negative self-intersection intersecting D negatively. But then D is effective iff $D - C$ is effective; one direction is obvious, while if D is effective, then C is a fixed component of D . The same argument applies in step 2, while in step 3, either σ is irreducible (so again the argument applies) or we have $D \cdot c < 0$ for some fixed component c of σ . We must have $c^2 \geq -2$, else c would have been removed in step 1; and similarly $c \neq e_m$. But then the classification of fixed components of effective simple roots lets us find a lexicographically smaller simple root having negative intersection with D .

Since we terminate at a nef divisor in the fundamental chamber, this algorithm also gives us an explicit decomposition of D as a nonnegative linear combination of C_α and $-d$ -curves. In this context, we note the following.

Proposition 4.4. *Let X be an anticanonical rational surface with $K_X^2 < 8$. Either every representative of $-K_X$ is integral, or some representative is a nonnegative linear combination of $-d$ -curves with $d \geq 1$.*

Proof. If some representative of $-K_X$ is reducible, then we can write

$$-K_X = D_1 + D_2 \tag{4.3}$$

for nonzero effective divisors C_1, C_2 , and it suffices to show that each D_i is linearly equivalent to a nonnegative linear combination of $-d$ -curves. Since D_1 is effective by assumption, we can write

$$D_1 \sim m(-K_X) + \sum_j c_j C_j \tag{4.4}$$

where each C_j is a $-d$ -curve for some $d \geq 1$ and all coefficients are nonnegative. This implies $D_1 + mK_X$ is effective, and thus $-D_2 = D_1 + K_X = D_1 + mK_X + (m-1)C_\alpha$ is effective, unless $m = 0$. In other words, D_1 has a decomposition as required. \square

Corollary 4.5. *Let X be an anticanonical rational surface, and suppose $K_X^2 \notin \{0, 1, 8, 9\}$. Then the effective monoid of X is generated by the integral curves of negative self-intersection.*

Proof. If $K_X^2 < 0$, then C_α has negative self-intersection, and is either integral or redundant. For $1 < K_X^2 < 8$, we note that $\Gamma(-K_X)$ corresponds to a codimension m subspace of $\Gamma(-K_{X_0})$. On a Hirzebruch surface, either every anticanonical curve is reducible (i.e., on F_d for $d > 2$), or the reducible anticanonical curves are codimension 2 subvariety of the 8-dimensional projective space of all anticanonical curves. We are imposing $m \leq 6$ linear conditions on this projective variety, and thus obtain a nonempty set of anticanonical curves on X which are reducible on X_0 and thus reducible on X . \square

Remark. Similarly, if $K_X^2 = 1$ but X has a -2 -curve, then some anticanonical curve is reducible. Also, in any case the anticanonical divisor is not needed to generate the *rational* effective cone when $K_X^2 = 1$, since then $-2K_X = (-2K_X - e_7) + e_7$ is a sum of effective divisors.

We can also adapt the algorithm to compute $h^0(\mathcal{L}(D))$ for an effective divisor. Indeed, every step of the algorithm removes a fixed component of D , and thus the resulting nef divisor D' has a natural isomorphism

$$H^0(\mathcal{L}(D')) \cong H^0(\mathcal{L}(D)). \quad (4.5)$$

Thus to compute the dimensions of effective linear systems, it remains only to compute the dimensions of linear systems corresponding to nef divisors in the fundamental chamber. So let D be such a divisor class and suppose $m = 0$ or $D \cdot e_m > 0$, since otherwise we may as well consider D as a divisor on X_{m-1} .

If $D \cdot C_\alpha > 0$, then it follows from [20, Thm. III.1(ab)] that $h^1(\mathcal{L}(D)) = 0$, and thus we can use Hirzebruch-Riemann-Roch to compute

$$h^0(\mathcal{L}(D)) = \chi(\mathcal{L}(D)) = \frac{D \cdot (D + C_\alpha)}{2} + 1. \quad (4.6)$$

This in particular holds whenever $m < 8$, since then either $D \cdot C_\alpha > 0$ or $D = 0$.

If $m \geq 8$ and $D \cdot C_\alpha = 0$, then from the proof of Proposition 4.2, we find that $D - C_\alpha$ is also nef. Now consider the short exact sequence

$$0 \rightarrow \mathcal{L}(D - C_\alpha) \rightarrow \mathcal{L}(D) \rightarrow \mathcal{L}(D)|_{C_\alpha} \rightarrow 0. \quad (4.7)$$

From [20, Thm. III.1(d)], we find that the natural inclusion

$$H^0(\mathcal{L}(D - C_\alpha)) \subset H^0(\mathcal{L}(D)) \quad (4.8)$$

is an isomorphism iff the line bundle $\mathcal{L}(D)|_{C_\alpha}$ is nontrivial. Since $h^0(\mathcal{O}_{C_\alpha}) = 1$, we conclude that

$$h^0(\mathcal{L}(D)) = \begin{cases} h^0(\mathcal{L}(D - C_\alpha)) + 1 & \mathcal{L}(D)|_{C_\alpha} \cong \mathcal{O}_{C_\alpha} \\ h^0(\mathcal{L}(D - C_\alpha)) & \text{otherwise.} \end{cases} \quad (4.9)$$

If $m > 8$, then $D - C_\alpha$ is a nef divisor with $(D - C_\alpha) \cdot C_\alpha > 0$, so we reduce to the previous case. If $m = 8$, then $D = rC_\alpha$ for some $r \geq 1$, and thus C_α must be nef. We deduce that either C_α is integral or every component of C_α is a -2 -curve. Moreover, the above recurrence tells us that in this case,

$$h^0(\mathcal{L}(rC_\alpha)) = \lfloor r/r' \rfloor + 1, \quad (4.10)$$

where r' is the order of the bundle $\mathcal{L}(C_\alpha)|_{C_\alpha}$ in the group $\text{Pic}(C_\alpha)$. (In particular, $h^0(\mathcal{L}(rC_\alpha)) = 1$ if this bundle is not torsion.)

Remark. If

$$D = ns + df - \sum_i r_i e_i \quad (4.11)$$

relative to some even blowdown structure, then

$$\chi(\mathcal{L}(D)) = (n+1)(d+1) - \sum_i \frac{r_i(r_i+1)}{2}. \quad (4.12)$$

This of course corresponds to the fact that $H^0(\mathcal{L}(D))$ is a subspace of $H^0(\mathcal{L}(ns + df))$ cut out by the appropriate number of linear conditions. (If X blows up m distinct points of X_0 , the conditions are simply that the curve have multiplicity r_i at the i -th point.) In principle, one could determine $h^0(\mathcal{L}(D))$ (and in particular test whether D is effective) using linear algebra, but the above approach scales better, and largely separates out the combinatorial influences from the algebraic influences.

4.3 Integral divisors

By Lemma II.6 and Theorem III.1 of [20], there is a relatively short list of possible ways that a nef class can fail to be generically integral. (The integral classes which are not nef are precisely the $-d$ -curves for $d \geq 1$ and the anticanonical divisor, when this is integral and has negative self-intersection, and we already know how to recognize those.) Although the description given there is purely geometric, it turns out to be easy enough to recognize the different cases in terms of the representation of the divisor in a fundamental chamber. Since this representation is unique, we can (and will) figure out how each case is represented by placing various geometrically motivated constraints on the blowdown structure, and checking that the result is in the fundamental chamber.

Remark. Note that in characteristic 0, “generically integral” and “generically smooth” are the same on an anticanonical rational surface: a generically integral divisor class on a rational surface has at most one base point, and if it does, meets C_α at that point with multiplicity 1. Bertini’s theorem implies that the generic representative is smooth away from the base point, and the intersection with C_α implies smoothness there.

Lemma 4.6. *Let D be a divisor on X , and suppose Γ is an even blowdown structure such that D is in the fundamental chamber. Then D is a pencil iff one of the following three cases occurs.*

- (a) $D = f$.
- (b) $D = 2s + 2f - e_1 - \cdots - e_7$, D is nef, and $\mathcal{L}(2s + 2f - e_1 - \cdots - e_7 - e_k)|_{C_\alpha} \not\cong \mathcal{O}_{C_\alpha}$ for $8 \leq k \leq m$.
- (c) $D = r(2s + 2f - e_1 - \cdots - e_8)$, where $\mathcal{L}(2s + 2f - e_1 - \cdots - e_8)|_{C_\alpha}$ is a line bundle of exact order r in $\text{Pic}(C_\alpha)$.

Proof. A pencil is certainly generically integral (lest X be reducible), so nef. Per [20, Lem. II.6], there are three possibilities: $D^2 = 0$, $D \cdot C_\alpha = 2$; $D^2 = D \cdot C_\alpha = 1$; or $D^2 = D \cdot C_\alpha = 0$.

In the first case, the generic fiber of D has arithmetic genus 0, so D is the class of a rational ruling. It follows that there exists a blowdown structure such that $D = f$, and we readily verify that f is in the fundamental chamber. (If the blowdown structure we end at is odd, simply perform an elementary transformation, and note that this preserves the meaning of f .)

For the case $D^2 = D \cdot C_\alpha = 0$, D gives a quasi-elliptic fibration of X , and we can choose a blowdown structure in which we first blow down any -1 -curves contained in fibers. After doing so, we end up at a relatively minimal quasi-elliptic surface, which must be X_8 for the blowdown

structure. The only isotropic vectors in $\text{Pic}(X_8)$ are the multiples of the canonical class, and thus $D = r(2s + 2f - e_1 - \cdots - e_8)$ for some r ; again, this is in the fundamental chamber. For this to be a pencil, it must not have any fixed component, so $\mathcal{L}(D)|_{C_\alpha} \cong \mathcal{O}_{C_\alpha}$ and $\mathcal{L}(-K_{X_8})|_{C_\alpha}$ has order dividing r . If the order strictly divides r , then $\mathcal{L}(D)$ will have more than 2 global sections.

For the case $D^2 = D \cdot C_\alpha = 1$, the linear system is again quasi-elliptic, now with a base point. The base point is on the anticanonical curve, namely the unique point such that

$$\mathcal{L}(D)|_{C_\alpha} \cong \mathcal{L}_{C_\alpha}(p). \quad (4.13)$$

The fibers of D transverse to C_α are either integral or contain a single -1 -curve, while the fiber not transverse to C_α contains C_α . The residual divisor $D - C_\alpha$ has arithmetic genus $1 - r$, where $r = C_\alpha^2 - 1$, and thus has at least r connected components, each of which has negative self-intersection (since it is orthogonal to a class of positive self-intersection). It follows that every component has self-intersection -1 and arithmetic genus 0, and thus contains a -1 -curve. We may thus choose a blowdown structure in which we first blow down those -1 -curves until eventually reaching X_7 and $D = 2s + 2f - e_1 - \cdots - e_7$. This is a pencil on X_7 precisely when it is nef, and remains a pencil on X as long as we never blow up the base point. \square

Proposition 4.7. *Suppose D is a nef divisor class with $D \cdot C_\alpha \geq 2$, and let Γ be an even blowdown structure for which D is in the fundamental chamber. Then D is generically integral unless $D = rf$ for some $r > 1$.*

Proof. By [20, Thm. III.1(a)], D is base point free, so is generically integral unless it is a strict multiple of a pencil. \square

The case $D \cdot C_\alpha = 0$, which is the most interesting for us in any event, is the next easiest case to handle. If $\mathcal{L}(D)|_{C_\alpha} \not\cong \mathcal{O}_{C_\alpha}$, then D can only be integral if $D = C_\alpha$ and C_α is integral ([20, Thm. III.1(d)]).

Theorem 4.8. *Let X be an anticanonical rational surface, let D be a nef divisor class on X such that $\mathcal{L}(D)|_{C_\alpha} \cong \mathcal{O}_{C_\alpha}$, and let Γ be an even blowdown structure such that D is in the fundamental chamber. Then D is generically integral unless one of the following two possibilities occurs.*

- (a) $D = r(2s + 2f - e_1 - \cdots - e_8)$, and $\mathcal{L}(2s + 2f - e_1 - \cdots - e_8)|_{C_\alpha}$ is a line bundle of order r' strictly dividing r . Then the generic representative of D is a disjoint union of r/r' curves of genus 1, of divisor class $r'(2s + 2f - e_1 - \cdots - e_8)$.
- (b) $D = r(2s + 2f - e_1 - \cdots - e_8) + e_8 - e_9$ with $r > 1$, and $\mathcal{L}(2s + 2f - e_1 - \cdots - e_8)|_{C_\alpha} \cong \mathcal{O}_{C_\alpha}$. Then the generic representative of D is the union of r divisors of class $2s + 2 - e_1 - \cdots - e_8$ (all of genus 1) and a -2 -curve of class $e_8 - e_9$.

Proof. D is generically integral unless it factors through a pencil or has a fixed component. The first case is precisely (a) above, by the classification of pencils. The second case is described in [20, Thm. III.1(c)]: D has a unique fixed component N which is a -2 -curve, and $D - N$ is a strict multiple of a pencil P with $P \cdot N = 1$. In particular, there exists a blowdown structure such that P is the total transform of some antipluricanonical pencil on X_8 , and “pluri” can be ruled out by the fact that $P \cdot N = 1$. Now, N cannot be contracted by the map $X \rightarrow X_8$, since P is still base point free on X_8 ; thus N is a rational curve, and since $N \cdot P = 1$, must be a -1 -curve. We can thus further insist that the map $X_8 \rightarrow X_7$ blows down N . Since N is a -2 -curve on X , the map $X \rightarrow X_8$ blows up a point of N exactly once, and we may insist that this is the first point blown up after reaching X_8 ; i.e., that N is already a -2 -curve on X_9 . But then $N = e_8 - e_9$, and D has the claimed form, which we verify is in the fundamental chamber. \square

Remark 1. For multiplicative Deligne-Simpson problems, a rather more complicated irreducibility condition was given in [11]. In particular, the above result gives a much stronger statement in the case of 3- and 4-matrix multiplicative Deligne-Simpson problems, and it is natural to wonder if a similarly strong result holds in general.

The remaining case $C_\alpha \cdot D = 1$ can be dealt with in one of two ways. The easiest is to blow up the intersection with C_α , and consider the strict transform D' of D on $X_{m+1} =: X'$, a divisor class which is generically disjoint from the new anticanonical curve. The above algorithms tell us how to determine the generic decomposition of such a divisor class: first use the algorithm for testing effectiveness to write it as a sum of (fixed) -2 -curves and a nef class in some fundamental chamber, then use the above result to decompose the latter class. The generic decomposition of D' on X' corresponds directly to the generic decomposition of D on X , since D' is a strict transform, and thus this procedure computes the generic decomposition of D .

We can also work out what the nonintegral cases look like in the fundamental chamber, again using a result of Harbourne, [20, Thm. III.1(b)]. We omit the details.

Proposition 4.9. *Suppose D is a nef divisor class on X such that $D \cdot C_\alpha = 1$, and let Γ be a blowdown structure for which D is in the fundamental chamber. Then D is generically integral except in the following two cases.*

- (a) *For some $1 \leq i \leq m$, $D \cdot e_i = 0$ and $\mathcal{L}(D - e_i)|_{C_\alpha} \cong \mathcal{O}_{C_\alpha}$. The fixed part of D is the total transform of the minimal such e_i .*
- (b) *$D = r(2s + 2f - e_1 - \cdots - e_8) + e_8$ and $\mathcal{L}(2s + 2f - e_1 - \cdots - e_8)|_{C_\alpha} \cong \mathcal{O}_{C_\alpha}$. The fixed part of D is the total transform of e_8 .*

We also mention a necessary condition for a divisor to be integral, related to the theory of Coxeter groups of Kac-Moody type. The convention there is to consider both “real” roots (i.e., roots in the usual sense) and “imaginary” roots. The latter are defined as integral vectors whose orbit intersects the fundamental chamber in a nonnegative linear combination of simple roots.

Proposition 4.10. *Any integral divisor D such that $D \cdot K_X = 0$ is a positive root (real or imaginary).*

We have already shown this for -2 -curves (i.e., that -2 -curves are positive real roots), while for nef curves it is a consequence of the following more general fact. Note that since $e_m \cdot C_\alpha = 1$, the fact that the simple roots are a basis of C_α^\perp implies that together with e_m , they form a basis of $\text{Pic}(X)$.

Proposition 4.11. *Let X be an anticanonical rational surface with $K_X^2 < 7$, and D a nef divisor class on X . Then for any blowdown structure on X , D is a nonnegative linear combination of the simple roots and e_m . In fact, if we write (for an even blowdown structure)*

$$D = a(s - f) + b(f - e_1 - e_2) + \sum_{1 \leq i < m} c_i(e_i - e_{i+1}) + c_m e_m, \quad (4.14)$$

then we have the inequalities

$$\begin{aligned} c_2 &\geq c_3 \geq \cdots \geq c_m \geq 0 \\ c_2 &\geq b \geq a \geq 0. \\ c_2 &\geq c_1 \geq 0 \end{aligned}$$

Proof. The divisor class e_i is effective for all i (since it is the total transform of a -1 -curve on X_i), and thus $D \cdot e_i \geq 0$. Taking $i \geq 3$, we conclude that

$$c_2 \geq c_3 \geq \cdots \geq c_m. \quad (4.15)$$

To see that $c_m \geq 0$, we note that

$$c_m = (c_m e_m) \cdot C_\alpha = D \cdot C_\alpha \geq 0. \quad (4.16)$$

Similarly, the classes $f + s - e_1 - e_2$, s , and f are effective on X_2 , thus on X , so taking inner products with D shows

$$c_2 \geq b \geq a \geq 0. \quad (4.17)$$

Finally, the effective classes $f + s - e_2$, $f + s - e_1$ tell us that

$$c_2 \geq c_1 \geq 0, \quad (4.18)$$

finishing the proof. \square

Remark 1. Of course, we can weaken the hypothesis “ X anticanonical” to “ $D \cdot K_X \leq 0$ ”, since the latter fact was the only way in which we used the anticanonical curve.

Remark 2. If D is in the fundamental chamber, the same sequences of coefficients will be convex.

5 Moduli of surfaces

5.1 General surfaces

One benefit of considering blowdown structures is that it makes the moduli problem of rational surfaces much better behaved. Of course, the standard approach of choosing an ample bundle also works, but obscures the symmetry of the situation; in contrast, as we have seen, working with blowdown structures gives us a (rational) action of the Coxeter group $W(E_{m+1})$.

To construct the moduli stack (an Artin stack) of rational surfaces with blowdown structures, we first need to construct the moduli stack of Hirzebruch surfaces. This is of course essentially just the moduli stack of rank 2 vector bundles on \mathbb{P}^1 , so is a standard construction, but it will be useful to keep in mind the details. (In particular, the construction we use is not the usual construction for the moduli stack of vector bundles; the extra structure we use to rigidify the moduli problem has a simpler interpretation as a structure on $\mathbb{P}(V)$.)

For any integer $d \geq 0$, we have

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(d+2), \mathcal{O}_{\mathbb{P}^1}) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(-d-2)) \cong k^{d+1}, \quad (5.1)$$

and thus the non-split extensions of $\mathcal{O}_{\mathbb{P}^1}(d+2)$ by $\mathcal{O}_{\mathbb{P}^1}$ are classified up to automorphisms of the two bundles by points of the corresponding \mathbb{P}^d . By a standard construction, this gives rise to a canonical extension

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^d}(1) \rightarrow V \rightarrow \mathcal{O}_{\mathbb{P}^1}(d+2) \boxtimes \mathcal{O}_{\mathbb{P}^d} \rightarrow 0 \quad (5.2)$$

of sheaves on $\mathbb{P}^1 \times \mathbb{P}^d$, each fiber of which is the corresponding non-split extension of $\mathcal{O}_{\mathbb{P}^1}(d+2)$ by $\mathcal{O}_{\mathbb{P}^1}$.

Let S_k denote the locally closed subspace of \mathbb{P}^d on which the fiber is isomorphic to $V_{d,k} := \mathcal{O}_{\mathbb{P}^1}(k+1) \oplus \mathcal{O}_{\mathbb{P}^1}(d+1-k)$; this gives a stratification of \mathbb{P}^d by S_k for $0 \leq k \leq d/2$. Each stratum can itself be identified as a moduli space of global sections of the $V_{d,k}$, namely the moduli space of

saturated global sections (i.e., generating a subbundle), modulo the action of $\text{Aut}(V_{d,k})$. Since the generic global section is saturated, we have

$$\dim(S_k) = \dim(\Gamma(V_{d,k})) - \dim(\text{Aut}(V_{d,k})) = d - \max(d - 2k - 1, 0); \quad (5.3)$$

in other words, $\dim(S_k) = 2k + 1$ except that $\dim(S_{d/2}) = d$.

Since $\dim(\Gamma(V_{d,k})) = d + 2$ is independent of k , this gives a flat map to the moduli problem of vector bundles of the form $V_{d,k}$ for $0 \leq k \leq d/2$. Taking the relative \mathbb{P} of the bundle gives a flat map to the moduli problem of Hirzebruch surfaces; since every Hirzebruch surface arises in this way for sufficiently large d , we find that the moduli problem of Hirzebruch surfaces is represented by an algebraic stack. (Note that if V, V' are nonsplit extensions, then $\text{Hom}(V, V')$ is given by a locally closed subset of global sections of V' , so $\text{Isom}(V, V')$ is indeed a scheme as required, and admits a quotient by \mathbb{G}_m to give $\text{Isom}_{\mathbb{P}^1}(\mathbb{P}(V), \mathbb{P}(V'))$.) Note that the stabilizers have the form $\mathbb{P} \text{Aut}(V_{d,k}) \rtimes \text{Aut}(\mathbb{P}^1)$.

This stack has two components (even and odd Hirzebruch surfaces), with generic fibers isomorphic to F_0 and F_1 respectively. In general, F_d has codimension $d - 1$ in the corresponding stack (simply compare automorphism group dimensions). Note that the smooth cover corresponding to V_d classifies pairs (X, σ) , where X is a Hirzebruch surface and $\sigma : \mathbb{P}^1 \rightarrow X$ is an embedding with $\text{im}(\sigma) \cdot f = 1$, $\text{im}(\sigma)^2 = d + 2$; the map to the moduli stack simply forgets σ .

To blow up, we proceed as in [19], based on an idea of Artin. (There, Harbourne constructed the moduli stack of blowups of \mathbb{P}^2 ; the extension to Hirzebruch surfaces is straightforward.) Now, let \mathcal{X}_0 denote the moduli stack of Hirzebruch surfaces, and let \mathcal{X}_1 denote the corresponding universal family of rational surfaces, with structure maps $\pi_0 : \mathcal{X}_1 \rightarrow \mathcal{X}_0$, $\rho : \mathcal{X}_1 \rightarrow \mathcal{X}_0 \times \mathbb{P}^1$. (This last is something of an abuse of notation; what we really mean is the \mathbb{P}^1 -bundle over \mathcal{X}_0 over which the vector bundles were constructed. Though this was a product over \mathbb{P}^d , we are quotienting by $\text{Aut}(\mathbb{P}^1)$.) Consider now the problem of classifying surfaces with $K_X^2 = 7$. Such a surface is uniquely determined by a pair (X_0, p) where X_0 is a Hirzebruch surface and $p \in X_0$ is a closed point. But points on a surface are classified by the universal surface, so that the moduli space of rational surfaces with blowdown structure such that $K_X^2 = 7$ is precisely \mathcal{X}_1 .

Now, extend this to a sequence of stacks \mathcal{X}_i and morphisms $\pi_i : \mathcal{X}_{i+1} \rightarrow \mathcal{X}_i$ for all $i \geq 0$ in the following way. Using the morphism π_{i-1} , we may construct the fiber product $\mathcal{X}_i \times_{\mathcal{X}_{i-1}} \mathcal{X}_i$ and then blow it up along the diagonal. Call the resulting blowup \mathcal{X}_{i+1} , and let π_i be the morphism induced by the first projection from the fiber product.

Proposition 5.1. *The stack \mathcal{X}_i represents the moduli problem of rational surfaces with blowdown structure and $K_X^2 = 8 - i$. The universal surface over this stack is $\pi_i : \mathcal{X}_{i+1} \rightarrow \mathcal{X}_i$, and the blowdown structure is induced by the maps*

$$\mathcal{X}_{i+1} \rightarrow \mathcal{X}_i \times_{\mathcal{X}_{i-1}} \mathcal{X}_i \rightarrow \mathcal{X}_i \times_{\mathcal{X}_{i-2}} \mathcal{X}_{i-1} \rightarrow \cdots \rightarrow \mathcal{X}_i \times_{\mathcal{X}_0} \mathcal{X}_1 \rightarrow \mathcal{X}_i \times \mathbb{P}^1 \quad (5.4)$$

In addition, for $m \geq 1$, each divisor class f, e_1, \dots, e_m is represented by a divisor on the universal surface, and there exists a line bundle of first Chern class $2s$.

Proof. This is a simple induction: \mathcal{X}_i is the universal surface over \mathcal{X}_{i-1} , so also the moduli space of pairs (X_{i-1}, p) . To obtain the universal surface over \mathcal{X}_i , we need to blow up p on the corresponding fiber, and this is precisely what blowing up the diagonal does for us.

The claim about divisors is clear for e_1, \dots, e_m , since these are just the total transforms of the corresponding exceptional curves. Similarly, $f - e_1$ is always a -1 -curve on X_1 , so gives rise to a divisor on the universal surface. To obtain a line bundle of class $2s$, take the bundle $\rho^! \mathcal{O}_{\mathbb{P}^1}^{-1}$. \square

Remark. For $m = 0$, there is a small difficulty having to do with the fact that the \mathbb{P}^1 could be twisted; once $m = 1$, we have guaranteed that the universal \mathbb{P}^1 in the construction has a point. (Of course, the anticanonical bundle on the base \mathbb{P}^1 is always defined, so lifts to a bundle of class $2f$.) Similarly, although the original construction on \mathbb{P}^d comes with a section of the Hirzebruch surface, \mathcal{X}_0 forgets that section, and the induced automorphisms can act nontrivially on the relative $\mathcal{O}(1)$.

Corollary 5.2. *The moduli stack of rational surfaces with blowdown structure has two irreducible components for each value of $K_X^2 \leq 8$, and each component is smooth of dimension $10 - 2K_X^2$.*

Proof. This is clearly true for \mathcal{X}_0 (since the generic Hirzebruch surface has a 6-dimensional automorphism group and the stack is covered by open substacks for which some $\mathbb{P}^k \times \mathrm{PGL}_2$ -bundle is isomorphic to \mathbb{P}^d), and each map π_i is smooth of relative dimension 2 (being a family of smooth projective surfaces). \square

Remark. Note that the two components are naturally isomorphic for $K_X^2 \leq 7$: just apply the standard elementary transformation. Also, the formula for the dimension holds for \mathbb{P}^2 as well, since $\dim \mathrm{Aut}(\mathbb{P}^2) = 8 = 2K_{\mathbb{P}^2}^2 - 10$.

The action of simple reflections on blowdown structures clearly extends to give birational automorphisms of these stacks (since each simple root is clearly generically ineffective). The action is undefined when the root is effective, leading us to wonder what those substacks look like. It turns out that any given positive root (simple or not) is effective on a closed substack of codimension 1. This is a special case of a more general fact about flat families of sheaves, which we will have occasion to use again.

Lemma 5.3. *Let $\pi : X \rightarrow S$ be a projective morphism of schemes, and suppose M is a coherent sheaf on X , flat over S . Suppose moreover that $R^p\pi_*M = 0$ for $p > 1$, and the fibers of M have Euler characteristic 0. Then the locus $T \subset S$ parametrizing fibers with global sections has codimension ≤ 1 everywhere. Moreover, where $T \subsetneq S$, it is a Cartier divisor.*

Proof. By [30], the derived direct image of M can be represented by a perfect complex on S starting in degree 0. Since the higher direct images vanish, we can replace the degree 1 term by the kernel of the map to the degree 2 term to obtain a two-term perfect complex. The Euler characteristic condition implies that the two terms have the same rank everywhere, and $R^1\pi_*M$ is supported on the zero locus of the determinant of the appropriate map, so has codimension ≤ 1 . Semicontinuity implies that the fibers of M have global sections precisely along the support of $R^1\pi_*M$. That this is a Cartier divisor follows from the construction of [30]: the determinant is the canonical global section of the bundle $\det \mathbf{R}\Gamma(M)^{-1}$. (Note that the reference only shows that the fibers have global sections on the zero locus of the canonical global section; the argument above gives the converse as well.) \square

Corollary 5.4. *For any positive (real) root of $W(E_{m+1})$, the corresponding divisor class is effective on a codimension 1 substack of \mathcal{X}_m .*

Proof. Indeed, a positive root has $D^2 = -2$, $D \cdot C_\alpha = 0$, and thus $\chi(\mathcal{L}(D)) = 0$. Since there exist surfaces for which no positive root is effective and surfaces for which every positive root is effective, the substack is nonempty, and not all of \mathcal{X}_m , so has codimension 1. \square

Remark. Similarly, that a surface in \mathcal{X}_9 admits some anticanonical curve is a codimension 1 condition.

Corollary 5.5. *The substack of rational surfaces containing a $-e$ -curve for some $e \geq 3$ is a countable union of closed substacks of codimension ≥ 2 .*

Proof. A $-d$ -curve on X_m for $d > 2$ is either the strict transform of a $-d$ -curve on X_0 or arises from a $-(d-1)$ curve on some X_k for $k < m$. We have already noted that X_0 contains -3 -curves (or worse) in codimension ≥ 2 . For the other case, we observe by induction (using the $d = 2$ case above) that X_k containing a -2 -curve is a countable union of codimension 1 conditions (since any effective root is a sum of -2 -curves, and in the absence of -3 -curves or worse, is uniquely effective). If we blow up a point of an effective root, the result will necessarily contain a -3 -curve, and any -3 -curve not already present must arise in this way. \square

Remark. Similarly, for $d > 3$, we have a $-d$ -curve or worse on a countable union of locally closed substacks of codimension $\geq d-1$. This fails to be closed since the -2 -curve we are turning into a $-d$ -curve could degenerate into a reducible effective root. So, for instance, the closed substacks corresponding to -4 -curves also contain configurations with two -3 -curves connected by a chain of -2 -curves.

If we try to extend the action of simple roots to the entire moduli space, we encounter a problem along these codimension ≥ 2 substacks. For simplicity in exhibiting the problem, we consider blowups of \mathbb{P}^2 . Define maps $p_1, p_2, p_3 : \mathbb{A}^1 \rightarrow \mathbb{P}^2$ by

$$p_1(u) = (0, 0), \quad p_2(u) = (u, 0), \quad p_3(u) = (0, u^2), \quad (5.5)$$

and define two family of surfaces parametrized by \mathbb{A}^1 : Y_u is the blowup of \mathbb{P}^2 in p_1, p_2 , then p_3 , while Z_u is the blowup of \mathbb{P}^2 in p_3, p_2 , then p_1 . (At each step, the maps not already used extend to the blowup, so give a well-defined point at which to blow up.) For $u \neq 0$, we have $Y_u \cong Z_u$, since the points are distinct, so the different blowups commute. On the other hand Y_0 is the blowup of F_1 in two *distinct* points of the -1 -curve, while Z_0 blows up the same point of the -1 -curve twice. But then $Y_0 \not\cong Z_0$, since Z_0 contains a -2 -curve, and Y_0 does not.

Since every rational surface locally looks like \mathbb{P}^2 , we see that there is no way to extend the action of S_3 on a three-fold blowup to include the surfaces where the three-fold blowup introduces a -3 -curve. This is essentially the only difficulty, however.

Theorem 5.6. *Let $\mathcal{X}_m^{\geq -2}$, $m \geq 2$, denote the stack parametrizing pairs (X, Γ) where X is a rational surface with $K_X^2 = 8 - m$ not containing any $-d$ -curves for $d > 2$ and Γ is a blowdown structure on X . There is a natural action of the Coxeter group $W(E_{m+1})$ on this stack, which for every simple root is given by the usual action on blowdown structures, where the root is ineffective.*

Proof. Since $m \geq 1$, we may use an elementary transformation to identify the two components of $\mathcal{X}_m^{\geq -2}$, and thus have both an odd and an even blowdown structure on X .

Since X has no $-d$ -curves for $d > 2$, an odd blowdown structure maps it to F_1 , so that we can proceed on to \mathbb{P}^2 . Moreover, every infinitely near point that gets blown up as we proceed to X is a jet. More precisely, the blowdown structure on X is determined by (a) a union F_{m+1} of jets on \mathbb{P}^2 and (b) a filtration F_i of the structure sheaf of this union such that each quotient is the structure sheaf of a point. When a given simple root $e_i - e_{i+1}$ is ineffective, the corresponding quotient F_{i+1}/F_{i-1} is supported on two distinct points, and the reflection makes the other choice of F_i . This extends immediately to the locus where the two points agree; the jet condition ensures that the degree 2 scheme parametrizing choices of F_i is separable. In particular, we find that the action of S_{m+1} extends to the full stack.

Similarly, an even blowdown structure corresponds to (a) a choice of F_0 or F_2 , (b) a union of jets on this surface (disjoint from the -2 -curve), and (c) a corresponding filtration. The same argument tells us that the corresponding S_m acts. Since the two subgroups cover the full set of simple roots, we obtain the desired action of $W(E_{m+1})$. (The braid relations hold because they hold generically.) \square

Remark 1. The stack $\mathcal{X}_m^{\geq -2}$ is not quite a substack of \mathcal{X}_m , in general, as it may be necessary to remove a countable infinity of closed substacks. For instance, we can obtain a -3 -curve in \mathcal{X}_9 by blowing up a point of any -2 -curve, and thus each of the infinitely many positive roots of E_{8+1} produces a different component we must remove.

Remark 2. Note that the simple reflections act trivially on the locus where the corresponding simple root is effective (thus a -2 -curve). Since that locus has codimension 1, we see that the simple reflections act *as reflections* on $\mathcal{X}_m^{\geq -2}$.

Remark 3. Note that although the group acts on $\mathcal{X}_m^{\geq -2}$, this action *cannot* extend to the universal surface. That is, the action preserves the isomorphism class of the surface, but the isomorphisms on generic fibers degenerate as we approach the bad fibers. The problem here is that the universal surface is itself a moduli space of surfaces, but those surfaces could contain -3 -curves; in other words, the generic isomorphisms degenerate precisely on the corresponding -2 -curves on the fiber.

Remark 4. This also helps quantify the sense in which the moduli stack of surfaces is badly behaved when we do not introduce the blowdown structure: even if we exclude $-d$ -curves for $d > 2$, it is the quotient of an Artin stack by a discrete group which is infinite when $K_X^2 \leq 0$.

Remark 5. Finally, the reader should be cautioned that although we have divisors corresponding to the standard bases of $\text{Pic}(X)$ (including s , since s is represented by a canonical divisor on X_0 in the odd case), these choices of divisor are not compatible with the action of the Coxeter group. (For instance, the reflection in $e_1 - e_2$ changes the representation of f from $(f - e_1) + e_1$ to $(f - e_2) + e_2$.) In particular, although the various line bundles are taken to isomorphic bundles under the group action, those isomorphisms are not canonical.

5.2 Anticanonical surfaces

When trying to extend the above construction to anticanonical surfaces, we encounter the difficulty that the dimension of the anticanonical linear system varies with the rational surface, and this variation depends in subtle ways on the configuration of $-d$ -curves on the surface with $d > 2$. It turns out that this is not too serious an obstruction to constructing the linear system, however.

Proposition 5.7. *Let L be a line bundle on a family X of smooth projective varieties over a d -dimensional locally Noetherian integral base S , such that on every fiber, $H^p(L) = 0$ for $p \geq 2$. Then the moduli functor $|L|$ defined by taking $|L|(T)$ to be the set of effective divisors on $X|_T$ with $\mathcal{L}(D) \cong L$ is represented by a locally projective scheme $|L|$ over S , of dimension at least $d + \chi(L) - 1$ everywhere locally. If the base is smooth and the dimension is equal to $d + \chi(L) - 1$, then $|L|$ is a local complete intersection.*

Proof. This certainly holds (and with equality for the dimension) when L is acyclic, since then Grauert's theorem tells us that the direct image of L is a vector bundle, and $|L|$ is just the corresponding projective bundle.

More generally, we may as well assume S is Noetherian and affine. In particular, there is an effective divisor D such that $L \otimes \mathcal{L}(D)$ is acyclic on every fiber, so that we may directly construct

$|L \otimes \mathcal{L}(D)|$. The long exact sequence of cohomology associated to

$$0 \rightarrow L \rightarrow L \otimes \mathcal{L}(D) \rightarrow L \otimes \mathcal{L}(D)|_D \rightarrow 0 \quad (5.6)$$

then tells us that $H^p(L \otimes \mathcal{L}(D)|_D) \cong H^{p+1}(L)$ on every fiber, and thus that $L \otimes \mathcal{L}(D)|_D$ is acyclic. The dual vector bundle to $\Gamma(L \otimes \mathcal{L}(D)|_D)$ then defines a system of equations on $|L \otimes \mathcal{L}(D)|$, cutting out those sections containing D . Call the resulting closed subscheme Y . Since Y is an intersection of $h^0(L \otimes \mathcal{L}(D)|_D)$ hypersurfaces, it has dimension at least

$$\dim |L \otimes \mathcal{L}(D)| - h^0(L \otimes \mathcal{L}(D)|_D) = d - 1 + \chi(L \otimes \mathcal{L}(D)) - \chi(L \otimes \mathcal{L}(D)|_D) = d - 1 + \chi(L) \quad (5.7)$$

everywhere locally. The restriction to Y of the universal divisor on $|L \otimes \mathcal{L}(D)|$ has D as a fixed locus, which we may subtract to obtain a universal divisor for $|L|$ as required.

The local complete intersection property follows from the fact that Y is obtained from a smooth scheme by intersecting the same number of hypersurfaces as its codimension. \square

Remark. Since automorphisms of L act trivially on $|L|$, we can apply this even when L is merely an isomorphism class of line bundles, with the one caveat being that this will make $|L|$ a family of Brauer-Severi varieties.

In particular, we may define the stack \mathcal{X}_m^α of anticanonical surfaces to be the linear system $|-K|$ on \mathcal{X}_m . Of course, we would like to know that this is irreducible, which will require some more work, as the Proposition only gives lower bounds on the dimension.

For Hirzebruch surfaces, things are not too difficult to control, as in each case we can write $-K = D_0 + D_1$ where D_0 is the divisor of fixed components and D_1 is acyclic:

$$\begin{aligned} F_0 : -K &= 0 + (2s + 2f) \\ F_1 : -K &= 0 + (2s + 3f) \\ F_2 : -K &= 0 + (2s + 2f) \\ F_{2d+1} : -K &= (s - df) + (s + (d + 3)f), \quad d \geq 1 \\ F_{2d} : -K &= (s - df) + (s + (d + 2)f), \quad d \geq 2 \end{aligned}$$

In particular, we find in each case that $|-K|$ is flat over the given locally closed substack, of relative dimension $\chi(D_1)$. This has the following curious effect: for all $k \geq 3$, the dimension of the stratum of \mathcal{X}_0^α corresponding to F_k is 0, while for F_0, F_1, F_2 the dimension is 2, 2, 1 respectively. (Recall that \mathcal{X}_0 itself has dimension -6 , since F_0 and F_1 have 6-dimensional automorphism groups.) Since \mathcal{X}_0^α has dimension at least 2 everywhere, we find that it has precisely two irreducible components, as we would have expected.

Remark. This is already a distinct departure from the general case, since now all $-d$ -curves for $d > 2$ are a codimension 2 phenomenon, not just the -3 -curves. It is clearer why this should be so for blowups: to obtain a $-d$ curve for $d > 2$, we need simply blow up a point of a $-(d - 1)$ curve. Since we already need to blow up a point of the anticanonical curve, this is a codimension 0 condition on the blowup! Thus really the question is when the anticanonical curve is reducible, and this is codimension 2 (either the curve is reducible on X_0 , or we must blow up a singular point).

Remark. Note that this moduli problem is not formally smooth. For instance, if $X = F_4$ and C_α contains s_{\min} with multiplicity 2, then the anticanonical section α extends to an anticanonical section on an open subset of \mathcal{X}_0 . This exhibits a subspace of the tangent space to (X, C_α) as a direct sum of the tangent space to X in \mathcal{X}_0 and the tangent space to α in $\mathbb{P}(H^0(\omega_X^{-1}))$. But this subspace is larger than the generic tangent space!

To understand \mathcal{X}_m^α in general, we may proceed by induction in m .

Theorem 5.8. *The moduli problem of classifying triples (X, C_α, Γ) , where X is a rational surface with $K_X^2 = 8 - m$, $C_\alpha \subset X$ is an anticanonical curve, and Γ is a blowdown structure, is represented by an Artin stack \mathcal{X}_m^α . This stack is a local complete intersection of dimension $m + 2$, with one irreducible component for each parity of blowdown structure, with both components integral. If $m \geq 1$, the two components are canonically isomorphic.*

Proof. We have already shown this for $m = 0$, and we also note that the anticanonical curve is generically smooth in that case. Now, the fibers of the natural forgetful map $\mathcal{X}_m^\alpha \rightarrow \mathcal{X}_{m-1}^\alpha$ are straightforward to determine: a point in the fiber just indicates which point of the anticanonical curve was blown up. In other words, \mathcal{X}_m^α is the universal anticanonical curve over \mathcal{X}_{m-1}^α . By induction, the latter has two irreducible components, both integral, and the generic fiber of either component has smooth anticanonical curve. In particular, we find that the fibers of \mathcal{X}_m^α over \mathcal{X}_{m-1}^α are all 1-dimensional, and generically integral, so that each component of \mathcal{X}_{m-1}^α has integral preimage. Moreover, we immediately find that $\dim(\mathcal{X}_m^\alpha) = m + \dim(\mathcal{X}_0^\alpha) = m + 2$ as required.

That the components are isomorphic for $m > 0$ follows by elementary transformation as before, and the local complete intersection property follows from the fact that $2m - 6 + \chi(-K_X) - 1 = m + 2$. \square

The above construction showing that $W(E_{m+1})$ cannot in general act in the presence of -3 -curves works equally well in the anticanonical case (as long as the surface we start with has $K_X^2 \geq 3$, so that there is an anticanonical curve containing any three points). Thus we introduce the substack $\mathcal{X}_m^{\alpha, \geq -2} \subset \mathcal{X}_m^\alpha$ as before, by excluding all $-d$ -curves for $d > 2$. This is actually a substack in this case, since as we noted above, once we bound the minimal section of the Hirzebruch surface, there are only finitely many possible configurations of components of the anticanonical curve. Of course, not having any $-d$ -curves for $d > 2$ is a very strong condition to impose on an anticanonical surface: in particular, for $K_X^2 < 0$, it forces C_α to be integral. (Indeed, otherwise the components of C_α are smooth rational curves, at least one of which has negative intersection with C_α , so is a $-d$ -curve for $d > 2$.)

Proposition 5.9. *For $0 \leq m < 8$, $\mathcal{X}_m^{\alpha, \geq -2}$ is a \mathbb{P}^{8-m} -bundle over $\mathcal{X}_m^{\geq -2}$, and thus has two smooth components, isomorphic if $m > 0$. For $m > 8$, $\mathcal{X}_m^{\alpha, \geq -2}$ can be identified with a closed substack of $\mathcal{X}_m^{\geq -2}$.*

Proof. For $m > 8$, the anticanonical curve is integral if it exists, and since it has negative self-intersection is rigid. Thus there is at most one anticanonical curve on a rational surface without $-d$ -curves for $d > 2$. We have seen that this is a closed codimension 1 condition for $m = 9$, while for $m > 9$, it combines the closed conditions that the image in $\mathcal{X}_{m-1}^{\geq -2}$ is in $\mathcal{X}_{m-1}^{\alpha, \geq -2}$ and that the point being blown up lies on the anticanonical curve.

For $m < 8$, the anticanonical divisor is nef on a surface without $-d$ -curves for $d > 2$, and since $C_\alpha^2 = 8 - m > 0$, the corresponding line bundle is acyclic. But then the linear system is a \mathbb{P}^{8-m} -bundle as required. \square

Remark. The case $m = 8$ is more subtle, as the surface could have a unique anticanonical curve, or could have a 1-parameter family of anticanonical curves (making it an elliptic surface with no multiple fibers).

In any case, since the choice of C_α is independent of the blowdown structure, the action of $W(E_{m+1})$ extends immediately to $\mathcal{X}_m^{\alpha, \geq -2}$. As before, the action does not extend to the universal

surface (it must act linearly on $\text{Pic}(X)$, so does not respect the effective cone), but it turns out that there is a strong sense in which it *does* act on the *line bundles* on the universal surface.

Given any vector $v \in \mathbb{Z}s + \mathbb{Z}f + \sum_i \mathbb{Z}e_i$, we have a corresponding line bundle \mathcal{L}_v on the universal surface over $\mathcal{X}_m^{\alpha, \geq -2}$. (In general, we only knew this when the coefficient of s was even, but the assumptions imply that an odd blowdown structure reaches F_1 , where s is canonically a divisor, and the claim for even blowdown structures follows by elementary transformation.) Of course, the space of global sections of this bundle can vary wildly with the surface, and can similarly vary if we replace v by wv for any element $w \in W(E_{m+1})$. These are essentially the same phenomenon, however. The main problem with the global sections of \mathcal{L}_v is that we can have sections of \mathcal{L}_v on a given surface that do not extend to neighboring surfaces in the moduli space. This can be fixed by taking the direct image sheaf rather than the fiberwise global sections. This can cause problems in general, however, which are characterized by the following result (a strong (albeit specialized) form of semicontinuity).

Lemma 5.10. *Let $\pi : X \rightarrow S$ be a projective morphism, and suppose that M is a sheaf on X , flat over S , such that every fiber of M has $H^p = 0$ for $p \geq 2$. Then for any sheaf N on S , we have isomorphisms*

$$\mathcal{T}or_{p+2}(R^1\pi_*M, N) \cong \mathcal{T}or_p(\pi_*M, N) \quad (5.8)$$

for $p > 0$, along with a short exact sequence

$$0 \rightarrow \mathcal{T}or_2(R^1\pi_*M, N) \rightarrow \pi_*M \otimes N \rightarrow \pi_*(M \otimes \pi^*N) \rightarrow \mathcal{T}or_1(R^1\pi_*M, N) \rightarrow 0 \quad (5.9)$$

and an isomorphism

$$R^1\pi_*M \otimes N \cong R^1\pi_*(M \otimes \pi^*N). \quad (5.10)$$

In particular, π_*M is flat iff $R^1\pi_*M$ has homological dimension ≤ 2 , the fibers of π_*M inject in the corresponding spaces of global sections of M iff $R^1\pi_*M$ has homological dimension ≤ 1 , and the injection is an isomorphism iff $R^1\pi_*M$ is flat.

Proof. As in the proof of Lemma 5.3, we find that $\mathbf{R}\pi_*M$ is represented by a two-term perfect complex on S . If $V^0 \rightarrow V^1$ is this complex, then we have a four-term exact sequence

$$0 \rightarrow \pi_*M \rightarrow V^0 \rightarrow V^1 \rightarrow R^1\pi_*M \rightarrow 0, \quad (5.11)$$

and thus any flat resolution of π_*M extends to a flat resolution of $R^1\pi_*M$. The claim follows upon tensoring this resolution with N and observing that

$$\mathbf{R}\pi_*M \otimes^{\mathbf{L}} N \cong \mathbf{R}\pi_*(M \otimes^{\mathbf{L}} \pi^*N) \cong \mathbf{R}\pi_*(M \otimes \pi^*N), \quad (5.12)$$

with the last isomorphism following from the fact that M is flat. \square

Remark. As an example, consider the divisor class $-2K_X$ on the moduli stack of anticanonical Hirzebruch surfaces. This acquires cohomology when the surface has a $-d$ -curve for any $d \geq 3$; since this locus has codimension 2, $R^1\pi_*(\omega_X^{-2})$ has homological dimension ≥ 2 . As a result, the fibers of the direct image sheaf do not inject in the spaces of global sections of the fibers. Similarly, the anticanonical bundle itself fails this criterion in the presence of a $-d$ -curve for $d \geq 4$.

By the last claim of the Lemma, we can compute $R^1\pi_*M$ fiberwise, and the main contribution comes from hypersurfaces: those where a given positive root becomes effective, and those where $\mathcal{L}_v|_{C_\alpha}$ has a global section. Near a generic point of such a hypersurface, we find that $\pi_*\mathcal{L}_v$ is flat and injects fiberwise in the space of global sections of \mathcal{L}_v , since $R^1\pi_*\mathcal{L}_v$ is a flat sheaf on a

hypersurface, so has homological dimension ≤ 1 . Although there could in principle be problems coming from intersections of the hypersurfaces, this at least suggests the following result; note that by the previous remark, we cannot allow worse than -2 -curves, even if we did not care about the $W(E_{m+1})$ action.

Theorem 5.11. *The direct image of any line bundle \mathcal{L}_v is a flat sheaf \mathcal{V}_v on $\mathcal{X}_m^{\alpha, \geq -2}$, and the action of $W(E_{m+1})$ extends to these sheaves. More precisely, for any element $w \in W(E_{m+1})$, we have an isomorphism*

$$w^* \mathcal{V}_v \cong \mathcal{V}_{wv}, \quad (5.13)$$

defined up to scalar multiplication, and the isomorphisms are compatible, again up to scalar multiplication. Moreover, the multiplication map

$$\mathcal{V}_v \times \mathcal{V}_{v'} \rightarrow \mathcal{V}_v \otimes \mathcal{V}_{v'} \rightarrow \mathcal{V}_{v+v'} \quad (5.14)$$

induced by

$$\mathcal{L}_v \otimes \mathcal{L}_{v'} \cong \mathcal{L}_{v+v'} \quad (5.15)$$

has no zero divisors.

Proof. First note that if v is not generically effective, then $\mathcal{V}_v = 0$, since then no global section of \mathcal{L}_v on a fiber can extend to an open substack of the moduli space. The generically effective divisors form a cone invariant under the action of $W(E_{m+1})$, so the various claims are immediate outside this cone. A generically effective divisor will have $v \cdot f \geq 0$, so $(-C_\alpha - v) \cdot f \leq -2$, and thus $-C_\alpha - v$ cannot be effective. We thus conclude that $H^2(\mathcal{L}_v) = 0$ for such a divisor, which is all we need to apply the Lemma.

It will suffice to show that whenever v is generically effective, the group acts and $R^1 \pi_* \mathcal{L}_v$ has homological dimension ≤ 1 . Indeed, this implies flatness of \mathcal{V}_v , as well as the fact that multiplication has no zero-divisors, the latter since the map

$$\Gamma(\mathcal{L}_v) \times \Gamma(\mathcal{L}_{v'}) \rightarrow \Gamma(\mathcal{L}_{v+v'}) \quad (5.16)$$

is injective on every fiber.

Now, using an elementary transformation as necessary, we may suppose our blowdown structure is odd and consider X as an $m+1$ -fold blowup of \mathbb{P}^2 . In the corresponding basis of $\text{Pic}(X)$, we have

$$v = nh - \sum_{0 \leq i \leq m} r_i e_i. \quad (5.17)$$

If $r_i = v \cdot e_i < 0$ for any i , then we have a short exact sequence

$$0 \rightarrow \mathcal{L}_{v-e_i} \rightarrow \mathcal{L}_v \rightarrow \mathcal{L}_v|_{e_i} \rightarrow 0. \quad (5.18)$$

On the generic fiber, the quotient is a sheaf of negative degree on the smooth rational curve e_i , and thus has no global sections; on the general fiber, the quotient has 1-dimensional support, so that the Lemma applies. We thus find that $\pi_*(\mathcal{L}_v|_{e_i}) = 0$ and (since that certainly injects!) that $R^1 \pi_*(\mathcal{L}_v|_{e_i})$ has homological dimension ≤ 1 . It follows that

$$\mathcal{V}_{v-e_i} \cong \mathcal{V}_v, \quad (5.19)$$

and $R^1 \pi_*(\mathcal{L}(v))$ has homological dimension ≤ 1 iff $R^1 \pi_*(\mathcal{L}(v - e_i))$ has homological dimension ≤ 1 . By induction, if we set

$$v' = nh - \sum_{0 \leq i \leq m} \max(r_i, 0) e_i, \quad (5.20)$$

then (since this operation respects the action of S_{m+1}) it suffices to prove the claim for v' .

Thus suppose $r_i \geq 0$ for $0 \leq i \leq m$, and consider the short exact sequence

$$0 \rightarrow \mathcal{L}_v \rightarrow \mathcal{L}_{nh} \rightarrow Q \rightarrow 0. \quad (5.21)$$

Since \mathcal{L}_{nh} is acyclic, \mathcal{V}_{nh} is flat, and we have exhibited \mathcal{V}_v as a subsheaf of this flat sheaf. Moreover, the fibers \mathcal{V}_v inject in $\Gamma(\mathcal{L}_v)$ iff they inject in the corresponding fibers of \mathcal{V}_{nh} .

Now, π_*Q is the kernel of a two-term perfect complex (since Q has 1-dimensional support), and is thus a subsheaf of a locally free sheaf. In particular, π_*Q is torsion-free, and thus the map $\mathcal{V}_v \rightarrow \mathcal{V}_{nh}$ is determined by its action on the generic fiber. This action is clearly S_{m+1} -covariant, and thus so is \mathcal{V}_v . Since the morphism $\mathcal{V}_v \rightarrow \mathcal{V}_{nh}$ determines the injectivity condition, we also conclude that if $R^1\pi_*\mathcal{L}_v$ has homological dimension 1, then so does $R^1\pi_*\mathcal{L}_{wv}$ for any $w \in S_{m+1}$.

A similar calculation with an even blowdown structure shows that the corresponding S_m acts on the bundles, and preserves the homological dimension condition. The one technicality is that the ambient bundle \mathcal{L}_{ns+df} need not be acyclic, but we can use S_{m+1} -invariance (conjugated by an elementary transformation) to assume $n \geq d$.

We thus now have full $W(E_{m+1})$ -covariance, so that it suffices to prove the homological dimension claim for v in the fundamental chamber. Of course, if $v = 0$, then $\mathcal{L}_0 = \mathcal{O}_X$, and the claim is obvious, so suppose $v \neq 0$. If $v \cdot C_\alpha > 0$, then \mathcal{L}_v is acyclic, and we are done. Otherwise, consider the short exact sequence

$$0 \rightarrow \mathcal{L}_{v+K} \rightarrow \mathcal{L}_v \rightarrow \mathcal{L}_v|_{C_\alpha} \rightarrow 0 \quad (5.22)$$

Generically, $\mathcal{L}_v|_{C_\alpha}$ is a nontrivial degree 0 sheaf on a smooth genus 1 curve, and thus we again find

$$\pi_*(\mathcal{L}_v|_{C_\alpha}) = 0 \quad (5.23)$$

and thus \mathcal{L}_v satisfies the homological dimension condition iff \mathcal{L}_{v-C_α} satisfies the homological dimension condition. \square

We should note a couple of things here. First, the argument shows that on a blowup of \mathbb{P}^2 , $\mathcal{V}_v \subset \mathcal{V}_{(v \cdot h)h}$ whenever $v \cdot h > 0$, with locally free quotient, and similarly $\mathcal{V}_v \subset \mathcal{V}_{(v \cdot f)s + (v \cdot s)f}$ relative to an even blowdown structure. This fact will guide the noncommutative construction in [44]; we will first construct noncommutative analogues of the ambient bundles, then impose suitable conditions on the generic fiber and use an analogue of the above argument to prove flatness.

Next, the result allows us to construct a flat family of categories with a nice action of $W_{E_{m+1}}$. The objects of the categories are the vectors $v \in \mathbb{Z}s + \mathbb{Z}f + \sum_i \mathbb{Z}e_i$, while the morphisms from v to v' are given by $\mathcal{V}_{v'-v}$, with the natural multiplication maps. The dimensions of the Hom spaces in this category are constant as we vary the choice of anticanonical surface, and the group acts in the obvious way. The construction of [44] will give a noncommutative deformation of this category, in the case C_α smooth; this will depend on one additional parameter (a point of $\text{Pic}^0(C_\alpha)$), but will have the same flatness properties. The Hom spaces of the deformation will be constructed as spaces of elliptic difference operators, and thus there is a close connection between modules over the deformed category and (symmetric elliptic) difference equations. (In particular, every symmetric elliptic difference equation will have a corresponding module over the deformation for $m = 0$.) This will be extended in [43] to a two-parameter deformation of the category with Hom spaces $S^n(\mathcal{V}_{v'-v})$.

In the case C_α integral, we can give a direct construction of the substack of surfaces with anticanonical curve isomorphic to C_α . (Presumably this can be extended to general curves, but it is unclear what the precise conditions will be.)

For any connected projective curve C (integral or not) of arithmetic genus 1, there is a natural moduli problem mapping flatly to the locally closed substack of \mathcal{X}_m^α where $C_\alpha \cong C$, namely the

problem of classifying triples (X, ϕ, Γ) where $\phi : C \rightarrow X$ embeds C as an anticanonical curve. Given such a triple, the restriction morphism $\phi^* : \text{Pic}(X) \rightarrow \text{Pic}(C)$ gives us a sequence of (isomorphism classes of) bundles $\phi^*(s)$, $\phi^*(f)$ and $\phi^*(e_i)$ for $1 \leq i \leq m$. The classes $\phi^*(e_i)$ have degree 1, $\phi^*(f)$ has degree 2, and $\phi^*(s)$ has degree 1 or 2 depending on whether Γ is odd or even.

Lemma 5.12. *The triple (X, ϕ, Γ) is determined up to isomorphism of X by the classes $\phi^*(s)$, $\phi^*(f)$, and $\phi^*(e_i)$, $1 \leq i \leq m$.*

Proof. We may assume X and C are defined over an algebraically closed field, so that the given classes actually correspond to line bundles. Since neither of $K_X + f$ or $-f$ is effective, we conclude that $H^0(\omega_X(f)) = H^2(\omega_X(f)) = 0$, and Euler characteristic considerations imply $H^1(\omega_X(f)) = 0$. It follows that we have a natural isomorphism $\Gamma(X, \mathcal{L}(f)) \rightarrow \Gamma(C, \phi^*(f))$, so that we can recover the induced map $\rho : C \rightarrow \mathbb{P}^1$ from $\phi^*(f)$ (up to PGL_2). Similarly, $\mathbf{R}\rho_*(\omega_X \otimes \mathcal{L}_s) = 0$, so that we have an isomorphism $\rho_*\mathcal{L}_s \cong \rho_*\phi^*(s)$ allowing us to recover X_0 as the projective bundle of $\rho_*(\phi^*(s))$, as expected.

Now, suppose we have reconstructed the surface X_k , and consider the direct image of $\phi^*(e_{k+1})$ on the corresponding anticanonical curve C_k . This can be identified with the direct image of the sheaf $\mathcal{L}(e_{k+1})|_{C_\alpha}$ on X , and thus by Corollary 6.7 of [42] fits into an exact sequence

$$0 \rightarrow \mathcal{O}_{C_k} \rightarrow \phi_{k*}\phi^*(e_{k+1}) \rightarrow \mathcal{O}_{p_{k+1}} \rightarrow 0 \quad (5.24)$$

where p_{k+1} is the point of C_k that gets blown up on X_{k+1} . We have

$$\Gamma(X_k, \phi_{k*}\phi^*(e_{k+1})) = \Gamma(X_{k+1}, \mathcal{L}(e_{k+1})|_{C_\alpha}). \quad (5.25)$$

Since e_{k+1} is a -1 -curve on X_{k+1} , we find that $\mathcal{L}(e_{k+1})$ is acyclic and uniquely effective, while

$$H^p(\mathcal{L}(e_{k+1} - C_\alpha)) \cong H^{2-p}(\mathcal{L}(-e_{k+1}))^* = 0. \quad (5.26)$$

It follows that $\phi_{k*}\phi^*(e_{k+1})$ is uniquely effective, so determines p_{k+1} and thus X_{k+1} . \square

For C integral, we readily see that any sequence of invertible sheaves of the correct degrees will give rise to a valid triple, and thus we can identify the moduli space of triples with the product

$$\text{Pic}^2(C) \times \text{Pic}^2(C) \times \text{Pic}^1(C)^m \quad \text{or} \quad \text{Pic}^1(C) \times \text{Pic}^2(C) \times \text{Pic}^1(C)^m, \quad (5.27)$$

depending on parity. This is a principal $\text{Aut}(C)$ -bundle over the corresponding substack of \mathcal{X}_m^α , and since the choice of embedding of C is independent of the choice of blowdown structure, the action of $W(E_{m+1})$ extends. Of course, this extension is just the obvious linear action! Note also that if we allow C to vary over the moduli space of smooth genus 1 curves, then this construction gives a dense open substack of \mathcal{X}_m^α .

For nonintegral curves, there will certainly be an additional constraint on the degree vectors of the invertible sheaves, since those degrees can be read off from the combinatorial type of (X, Γ) (i.e., the representations of the components of C_α in the standard basis). And, of course, there is the additional difficulty that the moduli stack of curves has many more pathologies once one allows nonintegral curves, especially since we do not want to impose any stability conditions. (For instance, there are reduced but reducible curves of arithmetic genus 1 that are not even Gorenstein.)

Along these lines, we note the following constraint on the anticanonical curve of a rational surface.

Lemma 5.13. *Let X be a rational surface, and suppose we can write $-K_X = A + B$ with A, B nonzero effective divisors. Then $H^1(\mathcal{O}_A) = H^1(\mathcal{O}_B) = 0$ and $A \cdot B = 2h^0(\mathcal{O}_A) = 2h^0(\mathcal{O}_B) > 0$.*

Proof. Since A and B are nonzero effective divisors, we find that $H^0(\mathcal{L}(-A)) = H^0(\mathcal{L}(-B)) = 0$, and thus by duality $H^2(\mathcal{L}(-A)) = H^2(\mathcal{L}(K_X + B)) = 0$. The long exact sequence associated to the natural presentation of \mathcal{O}_A has a piece

$$H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_A) \rightarrow H^2(\mathcal{L}(-A)), \quad (5.28)$$

and thus $H^1(\mathcal{O}_A) = 0$, with $H^1(\mathcal{O}_B) = 0$ following similarly. We thus have $h^0(\mathcal{O}_A) = \chi(\mathcal{O}_A) = \frac{1}{2}A \cdot (-K_X - A) = \frac{1}{2}A \cdot B$, so the remaining claim follows. \square

Remark. We also find $h^1(\mathcal{L}(-A)) = -\chi(\mathcal{L}(-A)) = \frac{1}{2}A \cdot B - 1$.

This has the following interesting consequence; this was established for anticanonical curves in \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$ in [6, Cor. 5.7], but given the above Lemma, the proof carries over directly.

Proposition 5.14. *Let (X, C_α) be an anticanonical rational surface. Then there is a natural action of the group scheme $\text{Pic}^0(C_\alpha)$ on C_α , which for an invertible sheaf $\mathcal{Q} \in \text{Pic}^0(C_\alpha)$ takes a point $p \in C_\alpha$ to the unique point p' such that $\mathcal{I}_{p'} \cong \mathcal{Q} \otimes \mathcal{I}_p$.*

Remark. Similarly, the proof of Proposition 5.9 and Lemma 5.10 op. cit. tells us that for any $\mathcal{Q} \in \text{Pic}^0(C_\alpha)$, the corresponding action $\tau_{\mathcal{Q}}$ fixes every singular point of C_α , stabilizes any irreducible component of C_α , and for any invertible sheaf \mathcal{L} one has

$$\tau_{\mathcal{Q}}^* \mathcal{L} \cong \mathcal{L} \otimes \mathcal{Q}^{-\chi(\mathcal{L})}. \quad (5.29)$$

The last part of Proposition 5.9 op. cit. suggests that if the component C occurs with multiplicity m , then $\tau_{\mathcal{Q}}$ restricts to the identity on $(m-1)C$.

5.3 Partitioning the moduli stack

Since the type and singularities of a difference or differential equation depends on how the anticanonical curve decomposes and interacts with the map to \mathbb{P}^1 , we would like to understand the corresponding decomposition of \mathcal{X}_m^α .

The simplest part of the decomposition is by whether the curve is smooth or of multiplicative or additive degeneration. More precisely, the “Picard type” of an anticanonical rational surface is defined to be one of the symbols e , $*$, or $+$, depending on whether $\text{Pic}^0(C_\alpha)$ is an elliptic curve, multiplicative group, or additive group.

Lemma 5.15. *The anticanonical rational surfaces of Picard type $*$ or $+$ (with blowdown structure) form an irreducible codimension 1 closed substack $\mathcal{X}_m^{\alpha, \bar{*}}$ of \mathcal{X}_m^α .*

Proof. If $m > 0$, then we may use the line bundle $\mathcal{L}(e_m)|_{C_\alpha}$ to induce a map from C_α to a weighted projective space with generators of degrees 1, 2, 3, and the image of C_α will be the Weierstrass model of $\text{Pic}^0(C_\alpha)$; then $\mathcal{X}_m^{\alpha, \bar{*}}$ is cut out by an invariant equation of degree 12.

For $m \leq 0$, we need merely observe that if we blow up a point of C_α , this will not change the Picard type, and thus the condition remains closed of codimension 1. \square

Remark. We could also use the nef divisor f on X_0 or h on \mathbb{P}^2 and reduce to known results on hyperelliptic curves of genus 1 or cubic plane curves. The Weierstrass case is particularly nice over $\mathbb{Z}[1/6]$, however, since then we obtain well-defined functions a_4, a_6 on the G_m -bundle over \mathcal{X}_m^α in which we have chosen a Poisson structure on X (since that corresponds to a choice of nonzero holomorphic differential). Then $\mathcal{X}_m^{\alpha, \bar{*}}$ is cut out in those coordinates by the equation $-64a_4^3 - 432a_6^2 = 0$.

The above works over \mathbb{Z} , but the additive case behaves differently in characteristic 2 and 3, and thus we have a slightly weaker result.

Lemma 5.16. *Over an algebraically closed field or a $\mathbb{Z}[1/6]$ -algebra, the anticanonical rational surfaces of Picard type $+$ (with blowdown structure) form an irreducible codimension 1 closed substack $\mathcal{X}_m^{\alpha,+}$ of $\mathcal{X}_m^{\alpha,*}$.*

Remark. Over $\mathbb{Z}[1/6]$, the additive Weierstrass curves are cut out from the full stack of Weierstrass curves by the equations $a_4 = a_6 = 0$, but over \mathbb{F}_3 the equations have degrees 2, 12 and over \mathbb{F}_2 they have degrees 1, 12; the same holds for hyperelliptic and cubic models.

One issue that arises with the stack $\mathcal{X}_m^{\alpha,*}$ is that it does not distinguish 0 and ∞ . For instance, in the irreducible case, there are two branches of C_α at the node, and the limit as we approach the two points is different. We can resolve this by taking a suitable double cover of the moduli space. Indeed, the Weierstrass model has a unique singular point, which we may take to be $(0, 0)$, giving a curve of the form

$$y^2 + a_1xy = x^3 + a_2x^2. \quad (5.30)$$

The two branches at ∞ are given by the roots of $b_1^2 + a_1b_1 - a_2$, and thus the double cover has a model of the form

$$y^2 + c_1xy = x^3, \quad (5.31)$$

where $c_1 = a_1 + 2b_1$ is uniquely determined if we have chosen a nonzero holomorphic differential on C_α . The double cover is ramified along $\mathcal{X}_m^{\alpha,+}$, and the equation for the corresponding substack is $c_1 = 0$, now valid over \mathbb{Z} . If C_α has multiplicative type with multiple components, then the double cover distinguishes the two branches at any given one of the nodes, and that identification can then be carried along the polygon.

We similarly define the “combinatorial type” of an anticanonical rational surface with blow-down structure to be the (multi)set of pairs $(C_i, m_i) \in \text{Pic}(X_m) \times \mathbb{Z}$, where C_i ranges over the components of the anticanonical curve and m_i is the corresponding multiplicity. Again, if we have two components with the same divisor class and multiplicity, we will usually want to distinguish them. We thus choose an ordering of the components, which gives us a somewhat different way to describe the type: if we have c components, then we have a morphism $\phi : \mathbb{Z}^c \rightarrow \text{Pic}(X_m)$ taking e_i to C_i , and a vector $\mu \in \mathbb{Z}^c$ giving the multiplicities, with the property that $\phi\mu = C_\alpha$.

The “type” of a rational surface is then defined to be the symbol $(c, \phi, \mu)_\sigma$ where (c, ϕ, μ) gives the combinatorial type and σ gives the Picard type. Note that most of the type the Picard type is determined by the combinatorial type: if $\max_i \mu_i > 1$, then the Picard type is $+$, while if $c > 3$ and $\max_i \mu_i = 1$, then the Picard type is $*$. So only the cases with $c \leq 3$ and no multiplicities have any ambiguity. (And, of course, the only case with Picard type e is $(1, (-K), (1))_e$.)

It is, in principle, straightforward to determine the set of possible types for any given m , as it is easy to write down a complete (though countably infinite) list for $m = 0$, and each point being blown up either lies on a single component or on a set of mutually intersecting components. One also has the following. Define the “dimension” of a type to be

$$\dim((c, \phi, \mu)_\sigma) = \begin{cases} m+1 & c=1, \sigma=* \\ m & c=1, \sigma=+ \\ m-1 & c=2, \mu=(11), \sigma=+ \\ m-2 & c=3, \mu=(111), \sigma=+ \\ 2m-6 + \sum_{1 \leq i \leq c} \phi_i \cdot (\phi_i - K)/2, & \text{otherwise.} \end{cases} \quad (5.32)$$

Lemma 5.17. *For any type $(c, \phi, \mu)_\sigma$, the anticanonical rational surfaces of that type form an irreducible Artin stack of dimension $\dim((c, \phi, \mu)_\sigma)$, which maps to a locally closed substack of the double cover of $\mathcal{X}^{\alpha, \bar{*}, m}$.*

Proof. We proceed by induction on m . For $m = -1$, the result is straightforward: since we have labelled the components, we may construct the stack inside the product of the linear systems on \mathbb{P}^2 , and the various conditions we want to impose or exclude are closed, with known codimension; subtracting $8 = \dim \operatorname{Aut}(\mathbb{P}^2)$ then gives the desired result for the dimension. For most cases with $m = 0$, we can proceed similarly; when the combinatorial type includes a section of nonpositive self-intersection $-d$, the surface must be F_d , while if it otherwise contains a component having negative intersection with $s - f$, it must be F_0 or F_1 . The only remaining cases are

$$(1, (2s + 2f), (1))_{e/*/+}, \quad \text{and} \quad (2, (s + f, s + f), (1, 1))_{*/+}.$$

The anticanonical curve on such a surface has a natural hyperelliptic model, and the surface itself is determined by this model together with a class in Pic^0 (which is trivial iff the surface is F_2), and thus the stack has dimension 1 more than the corresponding stack of hyperelliptic curves.

For $m > 0$, the type of X_m determines the type of X_{m-1} as well as the set of components the point being blown up lies on. Irreducibility is then almost immediate, as the fibers of the stack parametrizing X_m over the stack parametrizing X_{m-1} are either single points or open subsets of curves. (Here we use the fact that we have chosen branches at the singular points, and thus when blowing up a point on a surface of type $(2, \phi, (1, 1))_*$, we can distinguish the two points.)

It remains only to show that change in the dimension of the type is the same as the change in dimension of the classifying stack. If neither type falls into one of the exceptions in the dimension formula, then the change in the dimension of the type is indeed 1 or 0 depending on whether the point being blown up is on 1 or 2 components. It is then straightforward to deal with the exceptions case-by-case. \square

In particular, this tells us how many parameters a given type has, modulo the effect of automorphisms.

Of course, simply knowing the possible types of degenerations of surfaces (or of difference equations) is only part of the story: in general, we would like to understand the limiting relations between the different types. That is, we would like to know which types appear in the closure in \mathcal{X}_m^α of the locally closed substack corresponding to a particular type. Note that this is somewhat more than just a poset, as a given type may appear in multiple ways in the closure whenever there are ambiguities in the labelling. In particular, we expect the answer to be an ordered category rather than simply a poset. (There is a further issue, in that there are cases in small characteristic in which the intersection of two closures of types is not a union of types, see below.)

We have not been able to answer this question completely, but do have a couple of necessary conditions, the simpler of which is as follows. Define a (naïve) category structure on the set of types as follows. Let $(c_1, \phi_1, \mu_1)_{\sigma_1}$, $(c_2, \phi_2, \mu_2)_{\sigma_2}$ be a pair of types. If $\sigma_1 > \sigma_2$ relative to the order $e > * > +$, then there are no morphisms between the types, while otherwise a morphism is given by a linear transformation $\psi : \mathbb{Z}^{c_2} \rightarrow \mathbb{Z}^{c_1}$ with nonnegative coefficients such that $\phi_2 = \phi_1 \circ \psi$, $\psi(\mu_2) = \mu_1$. Note that any endomorphism in this category is an automorphism, so this is indeed an ordered category.

Lemma 5.18. *A flat morphism from a dvr to \mathcal{X}_m^α induces a morphism from the type of the (surface corresponding to the) special fiber to the type of the generic fiber.*

Proof. This is certainly true for the Picard type, so it suffices to consider the combinatorial type. Making a quasi-finite flat base change as necessary, we may assume that the irreducible components of the anticanonical curve on the generic fiber are geometrically irreducible, and then choose an ordering on the components to induce a well-defined combinatorial type. The corresponding family of anticanonical curves defines a divisor on the family of surfaces, not containing any fiber. As a result, the same applies to each irreducible component of that divisor. Those are in one-to-one correspondence with the components of the generic anticanonical curve, and thus (by restriction) induces a divisor on the special fiber corresponding to each component of the generic anticanonical curve. Each such divisor is contained in the anticanonical curve on the special fiber, and is thus a sum of geometric components of that curve. The linear transformation such that ψ_i is the linear combination of components corresponding to the restriction of C_i gives a morphism of combinatorial types as required. \square

We call a morphism in the category of types “effective” if it arises from a family over a dvr, and “strongly effective” if there is a family over a dvr such that the special fiber is the *generic* surface of the given type. It is easy to see that if ψ_1, ψ_2 are morphisms and ψ_1 is strongly effective, then $\psi_2 \circ \psi_1$ is (strongly) effective iff ψ_2 is (strongly) effective.

The one technique we have (other than producing an explicit deformation) for proving strong effectiveness is the following.

Proposition 5.19. *Let $\psi : T_1 \rightarrow T_2$ be a morphism such that for any factorization*

$$T_1 \xrightarrow{\psi_1} T' \xrightarrow{\psi_2} T_2$$

of ψ with $T' \not\cong T_2$, $\dim(T') < \dim(T_2)$. Then ψ is strongly effective.

Proof. Let $T_2 = (c, \psi, \mu)$, and let Y be the fiber product $\prod_i |\psi_i|$ of linear systems on X_m . There is a natural map from Y to \mathcal{X}_m^α given by taking the anticanonical curve to be $\sum_i \mu_i D_i$ where D_i is the fiber of $|\psi_i|$. It follows from Proposition 5.7 that Y has dimension at least $2m - 6 + \sum_{1 \leq i \leq c} \phi_i \cdot (\phi_i - K)/2$. Moreover, if the combinatorial type does not force the Picard type, then imposing the corresponding condition introduces one or two more hypersurfaces as appropriate. We thus obtain a stack Y_{T_2} of dimension at least $\dim(T_2)$ everywhere locally.

Now, it is easy to see that this construction defines a functor on the category of types, and thus we have a morphism $Y_{T_1} \rightarrow Y_{T'} \rightarrow Y_{T_2}$ for any factorization as described. Moreover, the moduli stack of surfaces of type precisely T embeds as an open substack of Y_T for each T .

Now, if ψ were not strongly effective, then the substack corresponding to T_1 would necessarily meet some component of Y_{T_2} not containing the substack corresponding to T_2 . Any such component has dimension at least $\dim(T_2)$, and thus the type T' of its generic fiber has at least that dimension. But the existence of surfaces of type T' in Y_{T_2} implies that there is a morphism $T' \rightarrow T_2$, and we thus obtain a factorization of precisely the form we excluded. \square

The following special cases are fairly straightforward, if occasionally tedious; we omit the details.

Corollary 5.20. *Any morphism between multiplicative types is strongly effective.*

Corollary 5.21. *Any morphism to a type (c, ϕ, μ) with $c \leq 3$, $\max_i \mu_i = 1$ is strongly effective.*

Corollary 5.22. *For $m \leq 1$, any morphism is strongly effective.*

Unfortunately, this is not the full story in general. To see this, consider the moduli stack of singular del Pezzo surfaces of degree 1. In our terms, this is the quotient stack $\mathcal{X}_7^{\alpha, \geq -2}/W(E_8)$,

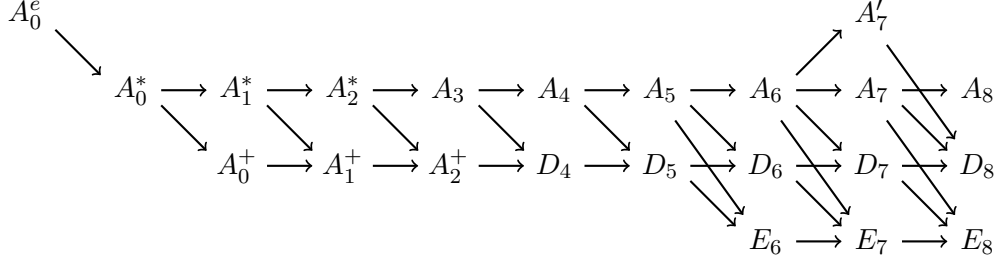


Figure 1: The natural stratification of $\mathcal{X}_7^{\alpha, \geq -2}/W(E_8)$ over $\mathbb{Z}[1/6]$.

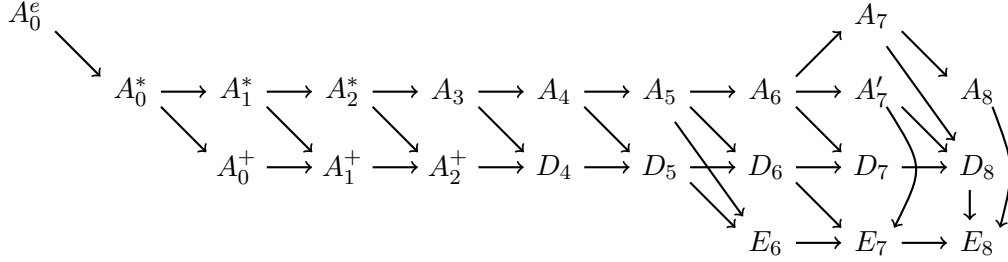


Figure 2: The naïve poset of types in $\mathcal{X}_7^{\alpha, \geq -2}/W(E_8)$.

and thus in particular inherits a decomposition from the one we have constructed on $\mathcal{X}_7^{\alpha, \geq 2}$. We can thus gain insight into the geometry of this decomposition by considering the quotient. In characteristic 0, a singular del Pezzo surface has an equation of the form

$$y^2 = x^3 + a_4(x, w)x + a_6(x, w) \quad (5.33)$$

where a_4, a_6 are homogeneous of the given degree. If we blow up the base point of the anticanonical linear system, we obtain an elliptic surface, and the type (mod $W(E_8)$) of the anticanonical curve $w = 0$ can be read off from the Kodaira symbol of the corresponding fiber of the elliptic surface. In particular, we can read off a parametrization of the surfaces of any given type from Tate's algorithm (see specifically the discussion in [51, §IV.9]), and then perform an elimination to determine the equations satisfied by the coefficients of a_4, a_6 in general. (There is a minor issue, in that the scheme corresponding to type I_8 is reducible, but it is easy enough to factor.) We thus obtain a collection of 22 ideals corresponding to the different possible special fibers, and it is easy enough to both determine the containment relations between the different ideals and to verify that the intersection of any two of the closures is a union of closures of types. We thus obtain a stratification of the moduli stack of del Pezzo surfaces *in characteristic 0*, which we may denote pictorially as in Figure 1.² (Here we specify the root system of the -2 -curves rather than the Kodaira symbol.) The open stratum A_0^e is $m + 2 = 9$ -dimensional, and each successive column decreases the dimension by 1. For surfaces with $K^2 = 0$, one obtains the same diagram, with dimensions increased by 1 (and affine rather than finite root systems), since $\mathcal{X}_8^{\alpha, \geq -2}$ is the universal $\text{Pic}^0(C_\alpha)$ over $\mathcal{X}_7^{\alpha, \geq -2}$, and the map on types coming from blowing down is clearly bijective.

Now, the components of the anticanonical curve of a del Pezzo surface of degree 1 all have negative self-intersection, so are rigid; as a result, if there is a morphism at all between two such types, it is unique. We find in particular that the corresponding subcategory of the category of types is simply the natural poset of root subsystems of E_8 (as extended by the Picard type), ordered

²These Hasse diagrams were adapted from the T_EX code for the analogous diagram in [25].

by inclusion. However, the resulting diagram (Figure 2) is slightly different (essentially from [50], except that we have added the missing arrow from A_7 to D_8 ; note that A_7 and A'_7 have swapped positions in the diagram to avoid arrows from the top row to the bottom row in each case). In particular, we find that although the combinatorics suggests that there should be degenerations

$$A'_7 \rightarrow E_7, \quad A'_7 \rightarrow E_8, \quad D_8 \rightarrow E_8, \quad A_8 \rightarrow E_8, \quad (5.34)$$

the corresponding morphisms are not effective. Note that the three cases corresponding to vertical arrows cannot possibly be *strongly* effective, since the dimensions of the corresponding substacks are the same.

In fact, the situation is even worse than this suggests: any surface of type E_8 in characteristic 3 can be obtained as the reduction mod 3 of a surface of type A_8 over a suitable 3-adic field. (Similar statements hold for the other three cases in characteristic 2.) Even if one restricts ones attention to equicharacteristic deformations, there are still issues: in characteristic 3, one can obtain *some* surfaces of type E_8 as limits from type A_8 . As a result, our decomposition of the moduli stack of del Pezzo surfaces is not a stratification in characteristic 3 or over \mathbb{Z} , as the closure of type A_8 meets type E_8 in a proper substack.

Indeed, over a field, a surface with an A_8 singularity has the form

$$y^2 + (a_{11}t + a_{10}u)xy + a_{30}u^3y = x^3, \quad (5.35)$$

up to changes of basis in x and y (but with no such changes of basis in t and u required). The discriminant of this surface has the form

$$c_1^2 t^3 u^9 + 3c_1 c_2 t^2 u^{10} + 3c_2^2 t u^{11} + c_3 u^{12} \quad (5.36)$$

for suitable functions c_1, c_2, c_3 of the parameters. The discriminant of an E_8 surface in characteristic $\notin \{2, 3\}$ has degree precisely 2 in t , but such a discriminant is not in the Zariski closure of the above set of discriminants, and thus there can be no degeneration from A_8 to E_8 in such cases. On the other hand, consider the del Pezzo surface

$$y^2 = x^3 - \frac{243t^2 - 54tu - u^2}{4}x^2 - \frac{3u^2(27t - 5u)}{2}x - \frac{tu^4(27t - 4u)}{4} \quad (5.37)$$

over the rationals. This has an A_8 singularity at $y = x = u = 0$, but modulo 3 becomes the surface

$$y^2 = x^3 + u^2x^2 + tu^5, \quad (5.38)$$

which now has an E_8 singularity at $y = x = u = 0$. A characteristic 3 surface with an A_8 singularity at $u = 0$ has discriminant of the form $u^9(at^3 + bu^3)$, and thus cannot degenerate to the above surface of discriminant $-tu^{11}$. However, the family of surfaces

$$y^2 = x^3 + v^3(tv - u)^2x^2 + u^2(u - vt)(u - v^3t)x + u^4t(u + v^3t) \quad (5.39)$$

over $\text{Spec}(\mathbb{F}_3[v])$ generically has an A_8 singularity at $u = 0$, but at $v = 0$ becomes the surface

$$y^2 = x^3 + u^4x + u^5t \quad (5.40)$$

with an E_8 singularity. Every smooth fiber of the latter surface has j -invariant 0, so is a supersingular curve. (There is one more isomorphism class of surfaces with an E_8 singularity, namely the quasielliptic surface $y^2 = x^3 + u^5t$, but this is a degeneration of the $j = 0$ case.)

Let us consider what a surface of type A_8 would look like if we considered it relative to a smooth anticanonical curve C' . To obtain type A_8 , two things must happen: all roots of A_8 must vanish in $\text{Pic}(C')$, but also every root *not* in the subsystem must *not* vanish. Indeed, if some root not in A_8 were to vanish, then every root of E_8 would vanish, and the surface would have type E_8 instead. Since the lattice Λ_{A_8} has index 3 in Λ_{E_8} , we see that the image of Λ_{E_8} in $\text{Pic}^0(C')$ is a 3-torsion subgroup. But in characteristic not 3, it is impossible to degenerate a nontrivial 3-torsion point to the identity. (In the 3-adic case, we can take the 3-torsion point to be in the kernel of reduction, while in the equicharacteristic case, we may take the special fiber of C' to be supersingular.)

Of course, the above argument is quite ad hoc, but it turns out that the obstruction generalizes. First, we note that the claim about degeneration of torsion points holds for curves of arbitrary genus.

Lemma 5.23. *Let R be a dvr with field of fractions K and residue field k . Let C_R be a smooth curve over R , and let L be a line bundle on C_R such that L_K has exact order r in $\text{Pic}(C_K)$. If $r \in R^*$, then L_k also has exact order r .*

Proof. We first note that $L_k^r \cong \mathcal{O}_{C_k}$, since this is a closed condition. But the assumption on r implies that $\text{Pic}^0(C_R)[r]$ is étale, and thus the closed subscheme $\text{Pic}^0(C_R)[d]$ is also open for any divisor $d|r$. It follows that $L_k^d \not\cong \mathcal{O}_{C_k}$ for every proper divisor $d|r$. \square

Remark. This also holds if R is equicharacteristic and C_k is ordinary, since then the complement of $\text{Pic}^0(C_R)[p^l]$ in $\text{Pic}^0(C_R)[p^{l+1}]$ is again both closed and open. The claim fails in the remaining cases with $\text{char}(k)|r$, however.

Now, a key fact about the curve C' we used above was that it was orthogonal to all of the roots in the relevant root systems. In particular, we can view C' as a curve on the surface obtained by contracting the -2 -curves. Moreover, C' is an *ample* divisor on that singular surface, and the claim boiled down to showing that the image of the Picard group of the minimal desingularization in C' contained a copy of $\Lambda_{A_8}^\perp/\Lambda_{A_8}$.

Note that in Lemma 5.23, we only consider the Picard groups of the two fibers; as a result, to apply the result, we need only understand surfaces over fields. Moreover, base changing to a finite extension of R has no effect on the order of L_K or L_k , and thus we can take a limit to a valuation ring in which both the residue field and the field of fractions are algebraically closed.

With that in mind, let Y be a normal surface over an algebraically closed field k , with minimal desingularization \tilde{Y} . To fix ideas, suppose for the moment that we have a rational map $\phi: Y \rightarrow \mathbb{P}^1$ such that the locus of indeterminacy is a single smooth point of Y and $\phi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is ample. Then blowing up the corresponding point of \tilde{Y} gives a surface \tilde{X} with a morphism to \mathbb{P}^1 . Now, let D be a divisor class on \tilde{X} such that the restriction of D to the generic fiber of \tilde{X} is principal. If we choose a function f with that divisor (which is uniquely determined modulo $k(\mathbb{P}^1)^*$), then $D - \text{div}(f)$ is certainly linearly equivalent to D , but now has trivial restriction to the generic fiber. This implies that $D - \text{div}(f)$ is supported on a finite set of fibers, and is thus a linear combination of components of fibers.

Now, suppose C is a component of a fiber. If C does not meet the exceptional curve of $\tilde{X} \rightarrow \tilde{Y}$, then its image in Y is disjoint from the generic fiber of ϕ . Since $\phi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is ample, this implies that the image of C must be a single point, and is thus one of the singular points of Y . We thus see that the components of fibers split into two classes: those which are contracted in Y , and those which meet the exceptional curve. Since the exceptional curve meets each fiber precisely once, we see that every fiber contains exactly one component meeting the exceptional curve.

Suppose C_1, \dots, C_n are the contracted components, and let C be any other component. If F is the fiber containing C , then $F - C$ is a sum of components not meeting the exceptional locus, and

thus we have

$$F - C \in \mathbb{Z}\langle C_1, \dots, C_n \rangle. \quad (5.41)$$

Now, suppose we are given a divisor class $D \in \text{Pic}(\tilde{Y})$ such that $rD \in \mathbb{Z}\langle C_1, \dots, C_n \rangle$ for some integer $r \geq 1$, and suppose that D has trivial restriction in the Picard group of the generic fiber of the pencil. Then the pullback of D to \tilde{X} is a linear combination of components of fibers, and therefore has an expression of the form

$$D \sim \sum_i a_i C_i + \phi^* Z \quad (5.42)$$

where $a_i \in \mathbb{Z}$ and Z is a divisor on \mathbb{P}^1 . But this implies that

$$(D - \sum_i a_i C_i) \cdot C_j = 0 \quad (5.43)$$

for all j . By Mumford's criterion for contractibility [34], the intersection form of C_1, \dots, C_n is negative definite, and we thus find that

$$D = \sum_i a_i C_i \in \mathbb{Z}\langle C_1, \dots, C_n \rangle. \quad (5.44)$$

In other words, the map from $\mathbb{Q}\langle C_1, \dots, C_n \rangle \cap \text{Pic}(\tilde{Y})$ to the Picard group of the generic fiber has kernel precisely $\mathbb{Z}\langle C_1, \dots, C_n \rangle$.

The main difficulty in applying the above argument in general is the requirement that the pencil have a single base point, as this requires in particular that $\langle C_1, \dots, C_n \rangle^\perp \subset \text{Pic}(\tilde{Y})$ contains smooth (and non-rigid) curves of self-intersection 1. We would thus like to extend the argument to deal with larger base loci. The difficulty, of course, is that fibers of the pencil can then have multiple components meeting the base locus.

Let Y, \tilde{Y} be as above, but now let \mathcal{L}_0 be an arbitrary very ample line bundle on Y . By Bertini's theorem, the corresponding linear system contains smooth curves, so let us fix such a curve C_0 . A further application of Bertini to C_0 lets us choose a curve C_1 in the linear system meeting C_0 in C_0^2 distinct points, and we can then choose C_2 meeting C_0 in the complement of $C_0 \cap C_1$, so that the linear system spanned by C_0, C_1, C_2 is base-point-free. Let P_{12} denote the pencil through C_1 and C_2 , and let P denote the pencil through C_0 and the generic point of P_{12} . Now, although P has base points, they are in general defined over an extension field of the field $k(\mathbb{P}^1)$ over which P is defined. In fact, we have the following.

Lemma 5.24. *The base points of P are defined over the separable closure of $k(\mathbb{P}^1)$, and form a single orbit under the action of the absolute Galois group $\text{Gal}(k(\mathbb{P}^1))$.*

Proof. That the splitting field of the base scheme of P is separable follows from the fact that $C_0 \cap C_1$ is reduced, and thus the same holds if we replace C_1 by the generic fiber of P_{12} . The linear system $P_{12}|_{C_0}$ is base-point-free and thus determines a morphism from C_0 to \mathbb{P}^1 . The base points of P are then just the preimage of the generic point under this morphism, and thus the base scheme is $\text{Spec}(k(C_0))$. Since $k(C_0)$ is a field, transitivity is immediate. \square

Now let \tilde{X} be obtained from \tilde{Y} by blowing up the base locus of P . If D is a divisor on \tilde{X} defined over k which is principal on the generic fiber of P , then, just as before, its preimage in \tilde{X} is linearly equivalent to an integer linear combination of components of fibers of P . Let D' be such a representative, and let σ be in the absolute Galois group of $k(\mathbb{P}^1)$. Then $\sigma D' - D' \sim \sigma D - D$ is

principal, but disjoint from the generic fiber, and is thus the divisor of a function pulled back from the base of the pencil. But then since $k(\mathbb{P}^1)$ has trivial Brauer group, we can add such a divisor to D' to make it Galois-stable. In particular, D' is a sum of Galois-orbits of components of fibers.

The key insight now is that the components which are contracted on Y are contained in k -rational fibers. If C is a component of such a fiber which is *not* contracted on Y , then its image in \tilde{Y} meets the base locus of P . But then by transitivity of the Galois group on the base points, we conclude that the image on \tilde{Y} of the sum of the Galois orbit of C actually *contains* the base locus. Then the resulting Galois stable curve is simply the strict transform of the corresponding fiber of P on Y , and may thus be expressed as a linear combination of the preimage of the fiber and the components of the preimage of the singular points.

Thus if C_i are the contracted components, we obtain an expression

$$D' = \sum_i a_i C_i + \sum_j F_j + \sum_l D_l \quad (5.45)$$

where each F_j is a fiber and each D_l is an effective divisor supported on a non-rational fiber. But then

$$D' \cdot C_j = \sum_i a_i C_i \cdot C_j, \quad (5.46)$$

and we can argue as before. We thus obtain the following result.

Proposition 5.25. *Let Y be a normal surface over an algebraically closed field k , with minimal desingularization \tilde{Y} and exceptional curves C_1, \dots, C_n . Let $\Lambda = \mathbb{Z}\langle C_1, \dots, C_n \rangle \subset \text{Pic}(\tilde{Y})$, and let $\Lambda^+ \subset \text{Pic}(\tilde{Y})$ be the subgroup consisting of line bundles such that $\mathcal{L}^r \in \Lambda$ for some $r \in \mathbb{Z}$ which is invertible in k . If C_0 is a smooth very ample curve on Y , then the restriction map $\text{Pic}(\tilde{Y}) \rightarrow \text{Pic}(C_0)$ induces an exact sequence*

$$0 \rightarrow \Lambda \rightarrow \Lambda^+ \rightarrow \text{Pic}(C_0) \quad (5.47)$$

Proof. The above argument shows that this holds for the generic curve in some pencil containing C_0 , and then the result follows by Lemma 5.23. \square

In the following result, note that by [5] any flat family of surfaces with sufficiently nice rational singularities admits a uniform minimal desingularization over an étale cover of the base. In our specific application, we start with a family of smooth surfaces, so there is no issue.

Proposition 5.26. *Let R be a dvr and $Y/\text{Spec}(R)$ be a projective scheme such that the fibers are normal rational surfaces, and suppose that $\tilde{Y}/\text{Spec}(R)$ is a fiberwise minimal desingularization of Y . Let C_1, \dots, C_n be the components of the exceptional locus of \tilde{Y}_k , and C'_1, \dots, C'_m the components of the exceptional locus of \tilde{Y}_K . Let a_{ij} be the multiplicity of C_j in the special fiber of the closure of C'_i , and let $\vec{a}_i \in \mathbb{Z}^n$ be the corresponding collection of vectors. Then the quotient*

$$(\mathbb{Q}\langle \vec{a}_1, \dots, \vec{a}_m \rangle \cap \mathbb{Z}^n) / (\mathbb{Z}\langle \vec{a}_1, \dots, \vec{a}_m \rangle) \quad (5.48)$$

is trivial if $\text{char}(k) = 0$ and a p -group if $\text{char}(k) = p$.

Proof. Both Y_k and Y_K have the property that their generic hyperplane section is smooth, and thus there is a hyperplane defined over R such that both fibers of the corresponding section C_0 are smooth. For any vector in $\mathbb{Q}\langle \vec{a}_1, \dots, \vec{a}_m \rangle \cap \mathbb{Z}^n$, let D be the corresponding linear combination of C_1, \dots, C_n . Since Y has rational fibers, the invertible sheaf $\mathcal{L}(D)$ extends uniquely from Y_k to Y , and some power of $\mathcal{L}(D)|_{Y_K}$ will have divisor class in $\mathbb{Z}\langle C'_1, \dots, C'_m \rangle$. The restriction $\mathcal{L}(D)|_{C_0}$ is trivial on the special fiber of C_0 , and thus by Lemma 5.23 is trivial (or has p -power order) on the generic fiber. The result follows from Proposition 5.25. \square

To apply this to anticanonical surfaces, we need to understand when configurations of anticanonical components can be contracted.

Lemma 5.27. *Let X be an anticanonical rational surface, and let C_1, \dots, C_l be a sequence of anticanonical components such that the intersection matrix of C_1, \dots, C_l is negative definite and not every component appears. Then the curve $\cup_i C_i$ is contractible.*

Proof. First note that if any of C_1, \dots, C_l are -1 -curves, then we can simply blow down that curve and apply the Lemma to X_{m-1} . We may thus assume that $C_i^2 \leq -2$ for $1 \leq i \leq l$. Now, let m_1, \dots, m_s be the multiplicities of the anticanonical components, and consider the divisor class $Z^+ = \sum_{1 \leq i \leq l} m_i C_i$. Then for $1 \leq i \leq l$,

$$Z^+ \cdot C_i = (C_\alpha - \sum_{l < j \leq s} m_j C_j) \cdot C_i = (2 + C_i^2) - \sum_{l < j \leq s} m_j (C_j \cdot C_i) \leq 0, \quad (5.49)$$

while

$$(Z^+)^2 = \sum_{1 \leq i \leq l} m_i (2 + C_i^2) - \sum_{1 \leq i \leq l, l < j \leq s} m_i m_j (C_j \cdot C_i) < 0, \quad (5.50)$$

since C_α is connected. It follows that the *fundamental cycle* [4] of $\cup_i C_i$ satisfies $Z = \sum_i r_i C_i$ with $1 \leq r_i \leq m_i$, and thus in particular both Z and $C_\alpha - Z$ are effective. It follows from Lemma 5.13 that every connected component of Z has arithmetic genus 0, and then [3, 4] tells us that every connected component of $\cup_i C_i$ contracts to a rational singularity. The result essentially follows immediately. (Artin assumes the configuration is connected, but the proof carries over.) \square

Corollary 5.28. *Let $\psi : T_1 \rightarrow T_2$ be a morphism of types, suppose C_1, \dots, C_l is a contractible configuration of anticanonical components of T_1 , let $\Lambda_1 \subset \text{Pic}(X_m)$ be the corresponding sublattice, and let $\Lambda_2 \subset \Lambda_1$ be the sublattice coming from those components of T_2 which are supported on C_1, \dots, C_l (relative to ψ). If Λ_1/Λ_2 has torsion of degree prime to the characteristic, then ψ is ineffective.*

This in particular explains the obstructions for del Pezzo surfaces of degree 1 (and corresponding obstructions for rational surfaces with $K^2 = 0$ and no $-d$ -curves with $d > 2$, the subject of [50]), as well as giving obstructions in other cases. For instance, let T_2 be the type of (quartic del Pezzo) surfaces with anticanonical curve decomposition:

$$C_\alpha = (s - e_1 - e_4) + (f - e_1 - e_2) + (s - e_2 - e_3) + (f - e_3 - e_4) + (e_1) + (e_2) + (e_3) + (e_4), \quad (5.51)$$

and let T_1 correspond to surfaces with

$$C_\alpha = 2(s - f) + 4(f - e_1 - e_2) + 3(e_1 - e_2) + 6(e_2 - e_3) + 5(e_3 - e_4) + 4(e_4). \quad (5.52)$$

Since the components of T_1 are linearly independent, it is easy to verify that there is a morphism $\psi : T_1 \rightarrow T_2$. On the other hand, the only components of T_2 supported on the complement of e_4 are the first four. The corresponding sublattice of \mathbb{Z}^5 is not saturated (it contains $2e_2 - 2e_4$ but not $e_2 - e_4$), and thus such a morphism can only correspond to a degeneration when the special fiber has characteristic 2. Here we should note that if we blow up a generic point on each component of the generic fiber (and then blowup a corresponding point downstairs), then there are no subsets of the anticanonical components to which the obstruction applies. Thus to apply this obstruction fully, one must in principle consider not only the types themselves but also all possible blowdowns. It is unclear if the resulting obstruction is effective. . .

For surfaces with $K^2 = 0$, although the computer calculation only showed that Figure 1 was valid in characteristic 0, it is not too hard to extend it to $\mathbb{Z}[1/6]$. If E is a fixed elliptic curve, then del Pezzo surfaces (with blowdown structure) with anticanonical curve E are classified by maps $\phi \in \text{Hom}(\Lambda_{E_8}, E)$. If $R \subset E_8$ is an indecomposable subsystem, then there is an anticanonical curve of type R if every root in $\ker \phi$ is contained in R . In particular, if Λ_R is saturated in Λ_{E_8} , then the generic point in $\text{Hom}(\Lambda_{E_8}/\Lambda_R, E)$ will give rise to such a surface. Moreover, we may verify by direct computation that the generic surface with such an anticanonical fiber has non-constant j -invariant, at which point dimension considerations tell us that the generic surface arises from a map $\text{Hom}(\Lambda_{E_8}/\Lambda_R, E)$. It follows easily that if $R \subset R'$ with Λ_R saturated, then the corresponding degeneration is strongly effective in any characteristic. In particular, every arrow in Figure 1 remains strongly effective over any field.

One is thus left to consider degenerations from A'_7 , A_8 , D_8 . Since $\Lambda_{E_8}/\Lambda_{D_8} \cong \mathbb{Z}/2\mathbb{Z}$, and $\Lambda_{E_8}/\Lambda_{A_8} \cong \mathbb{Z}/3\mathbb{Z}$, such surfaces are classified by points of order precisely 2 or 3 on E as appropriate, and thus over $\mathbb{Z}[1/6]$ cannot be degenerated. Similarly, A'_7 surfaces with a fiber isomorphic to E are labelled by a 2-torsion point and a point in E , and thus degenerate to the surface of type D_8 labelled by the same 2-torsion point, but not to surfaces of type E_7 or E_8 over $\mathbb{Z}[1/6]$.

This also makes it relatively straightforward to construct the exotic degenerations in small characteristic, as one need simply construct an appropriate family of curves E with a 2- or 3-torsion point degenerating to the identity. (This is how the above examples of $A_8 \rightarrow E_8$ degenerations were produced.)

5.4 Surfaces and singularities of equations

Now that we have introduced the decomposition of the moduli stack into types, it is natural to ask what a given type implies about the corresponding difference/differential equations. The type of a blown up surface translates into information about how the direct image of a sheaf with given Chern class meets the anticanonical curve on F_2 , and we argued above that this controls the singularities of the equation. Though it is clear that the sheaf must meet the anticanonical curve on F_2 at any singular point of the equation, and that the equation is singular at any point in the support of the intersection, we have not so far been clear about how the local behavior of the equation at the singularity is recorded in the sheaf structure of the intersection. For any given equation, of course, one can simply turn it into a sheaf and keep track of the different blowups; however, when classifying degenerations of a given elliptic scenario, one generally obtains simply a list of possible types of surface, and would then like to translate that into information about the singularities of the equation.

We restrict our attention to the case that the sheaf on F_2 can be separated from the anticanonical curve by a sequence of blowups and minimal lifts. This is a relatively mild restriction, since by [42] any sheaf which is transverse to the anticanonical curve can be put into that form by a sequence of “pseudo-twists”, which on difference equations correspond to canonical gauge transformations. In particular, the cases which cannot be so separated necessarily involve “apparent” singularities, in that there is some gauge transformation making its singularities milder. We similarly assume that the support of the sheaf does not contain a fiber, as again this leads to apparent singularities.

More precisely, since singularities are local in nature, we are interested in singularities at some fixed point $x \in \mathbb{P}^1$, and only need to be able to separate the sheaf from the anticanonical curve in a neighborhood of that fiber. We are thus given a blowup of F_2 in which every point blown up is on the same fiber, and a sheaf M on that blowup such that $x \notin \rho(\pi(\text{supp}(M) \cap C_\alpha))$. If M has first Chern class $ds + d'f - r_1e_1 - \cdots - r_me_m$, then the intersection of π_*M with the anticanonical curve near $\rho^{-1}x$ is independent of d' , and thus the question is to translate the remaining data into local

information about the equation. Equivalently, we are given the intersection numbers of $c_1(M)$ with the various components of the preimage of x .

Again, since singularities are local, we should actually replace the surface by a formal neighborhood of the fiber. Helpfully, the intersection numbers are still well-defined even when M is only a sheaf on the formal completion of the fiber, since the various components are still proper curves. Now, on the formal neighborhood, M has a natural direct sum decomposition, with one summand for each point of intersection. Moreover, at each point of intersection, M has finitely many branches. It follows that M is an extension of invertible sheaves on unibranched curves, which since we are working in a formal neighborhood, are isomorphic to the structure sheaves of their supports. Moreover, the cases in which the branch is tangent to the relevant exceptional component are limits of cases in which there are multiple branches, and thus we reduce (up to issues of limits not changing the restriction to C_α) to the case that the branch meets the exceptional component simply. Thus, up to the extension and limit problem, we reduce to the case that M is the structure sheaf of the image of an appropriate map from $\text{Spec}(k[[t]])$ to the formal completion of the fiber.

When applying this to the problem of classifying degenerate equations, there is one complication, however: at the elliptic level, we allowed *twisted* equations, and thus the sheaf we are given may not be on F_2 . Although in principle one could deal with this by working with connections twisted by suitable line bundles, this will not give equations in the form one usually wants to consider. To obtain a difference or differential equation with rational function coefficients, we need to choose a divisor representing the twisting line bundle. This is equivalent to choosing a section of the Hirzebruch surface which is transverse to the anticanonical curve. Given such a choice, we can translate sheaves into equations by first performing a sequence of elementary transformations to make the chosen section disjoint from the anticanonical curve, at which point we will be in the F_2 scenario where the translation is as described above. It is straightforward to see that changing the chosen section has the effect of gauging by a *scalar* function, the solution of an appropriate first-order equation. That is, if M is a sheaf on F_2 and we apply the above translation using a section s' other than the -2 -curve, then the result is to gauge by the solution of the equation corresponding to $\mathcal{O}_{s'}$. In particular, once we have understood singularities in the F_2 case, we will be able to understand the general case, up to an overall scalar gauge transformation freedom (which we can also understand).

Note also that once we have restricted M to a single branch (or even just a single point of intersection), we may feel free to blow down any component of the fiber which is a -1 -curve not meeting M . After doing so, we obtain a surface such that every point we blow up after the first is a point on the previous exceptional curve, as otherwise the final exceptional locus would contain disjoint -1 -curves. Moreover, any time we blow up a point which is on just one anticanonical component, the type of surface that results is not affected by the choice of point. In particular, once the most recent -1 -curve is not an anticanonical component, we should stop, as future blowups of points on that curve will give the same type as more general blowups on the anticanonical component it meets. We thus restrict our attention to types of surfaces in which every blowup but the first blows up a point of an exceptional component and every -1 -curve but the last is an anticanonical component. Call such a type “minimal”. Note that we may also insist that our curve be disjoint from the strict transform of the fiber, as again otherwise we can deform it to one meeting a more general point of the relevant -1 -curve.

In this way, we reduce to the following problems:

- (1) Given a minimal type, what do the resulting maps from $\text{Spec}(k[[t]])$ to F_2 look like? (And can this be inverted?)
- (2) Given a map from $\text{Spec}(k[[t]])$ to F_2 coming from a minimal type, what is the local structure

of the corresponding equation?

- (3) What is the effect of taking limits from the typical case of a minimal type to more special cases?

We will not consider (3) here (except in Section 8 for the elliptic case), as for most applications it is the typical behavior that matters. Presumably, the only effect of taking such limits is to make changes on the order of the $o()$ terms below.

Of the remaining cases, the simpler to deal with is (2), subject to the question of which maps come from minimal types. Supposing for the moment that the fiber of interest is $x = \infty$ (which will be the main interesting case in any event), then the map to F_2 takes the form $(y, x, w) = (Y(t), 1, W(t))$ for suitable power series Y and W . Moreover, $W(t)$ cannot be identically 0, since then the image would be contained in the fiber. The key point now is that we may view the formal neighborhood of the fiber as a \mathbb{P}^1 -bundle over $\text{Spec}(k[[w]])$, and perform our usual calculations to convert between sheaves on the \mathbb{P}^1 -bundle and morphisms on the anticanonical curve. In our case, the sheaf is the structure sheaf of an affine curve, and is thus simply $k[[t]]$ viewed as a $k[[w]]$ -module via the map $w \mapsto W(t)$. Since this is a totally ramified extension, $k[[t]]$ is a free $k[[w]]$ -module with basis $(1, t, \dots, t^{\text{ord}(W)-1})$, and the action M_t of multiplication by t in this basis is the companion matrix of the minimal polynomial of t over $k[[w]]$. The associated morphism of vector bundles on F_2 is then simply $B := y - Y(M_t) \in \text{Mat}_n(k[[w]][y])$. Note that in characteristic 0, we may change variables so that $W(t) = t^{\text{ord}(W)}$.

If the point $(0, 1, 0)$ we are blowing up is a typical (i.e., not ramified for the involution) smooth point of the fiber, then the condition that the spectral curve be disjoint from the strict transform of the fiber forces $\text{ord}(W) \leq \text{ord}(Y)$. Indeed, the spectral curve contains $(0, 1, 0)$ with multiplicity $\min(\text{ord}(W), \text{ord}(Y))$ and meets the fiber with multiplicity $\text{ord}(W)$, so after blowing up meets $f - e_1$ with multiplicity $\text{ord}(W) - \min(\text{ord}(W), \text{ord}(Y))$, which must vanish. We then find that the spectral curve meets the original anticanonical curve with multiplicity at least $\text{ord}(W)$, which must therefore be 1. This, of course, translates directly to the case of a simple singularity: A vanishes to order 1 at $(0, 1, 0)$ and has a simple pole at the image under the involution.

The next simplest case to consider is actually the differential case, where we assume characteristic 0. Here the anticanonical curve is $y^2 = 0$, and the reduced curve has a natural parametrization $(0, 1, 1/z)$, so that in terms of the parameter we have $z = W(t) = t^{-\text{ord}(W)}$. We again find that to avoid the strict transform of the fiber, we must have $\text{ord}(W) \leq \text{ord}(Y)$. The resulting Higgs bundle then has the form $A(z)^{-t} = -Y(M_t)|_{w=1/z}$. Note that we may view the resulting equation as one over the field of Puiseux series; if $W(t) = t^a$, $\text{ord}(Y(t)) = b$, then the equation has the form

$$\frac{v'(z)}{v(z)} = z^{b/a} \sum_{0 \leq l} c_l z^{-l/a-2}, \quad (5.53)$$

where $t^b/Y(t) = \sum_{0 \leq l} c_l t^l$. (Here the factor of z^{-2} comes from the fact that we need to work relative to a differential which is holomorphic at $z = \infty$.) This should be compared with the classification of the local behavior of differential equations [12], though we will see below that the combinatorics of the surface actually encodes more subtle information about which coefficients are nonzero.

The nonsymmetric q -difference case is the next simplest, as the anticanonical curve is $y^2 - xwy = 0$, and thus we can easily parametrize both branches:

$$(y, x, w) = (0, 1, 1/z) \quad \text{and} \quad (y, x, w) = (z, 1, z); \quad (5.54)$$

here we have parametrized so that the singular point is ∞ on one branch and 0 on the other branch. We again have $\text{ord}(W) \leq \text{ord}(Y)$, and the equation has

$$A(t) = \frac{Y(t)}{Y(t) - W(t)} \Big|_{t \rightarrow M_t}. \quad (5.55)$$

(Here we use the fact that we only care about the conjugacy class of M_t over $k[[w]] = k[[1/z]]$, and M_t and its transpose are conjugate.)

The nonsymmetric difference case is analogous, with the parametrizations now being

$$(y, x, w) = (0, 1, 1/z) \quad \text{and} \quad (y, x, w) = (z^2, 1, z), \quad (5.56)$$

of the anticanonical curve $y^2 - w^2y = 0$. Again, $\text{ord}(W) \leq \text{ord}(Y)$, and the equation is

$$A(t) = \frac{Y(t)}{Y(t) - W(t)^2} \Big|_{t \rightarrow M_t}. \quad (5.57)$$

The symmetric q -difference case is only slightly more complicated. The two branches on the anticanonical curve of the corresponding double cover correspond to the parametrizations

$$(y, x, w) = (\eta z/(z^2 + \eta)^2, 1, z/(z^2 + \eta)) \quad \text{and} \quad (y, x, w) = (z^3/(z^2 + \eta)^2, 1, z/(z^2 + \eta)) \quad (5.58)$$

of the irreducible anticanonical curve $y^2 - xwy + \eta w^4 = 0$. The key point is that although w is a quadratic function of z , the corresponding extension of $k[[w]]$ is unramified. We may thus let $Z(t)$ be the non-holomorphic solution of

$$Z(t) + \frac{\eta}{Z(t)} = \frac{1}{W(t)} \quad (5.59)$$

in $k((t))$ and view M_t as a matrix over $k[[1/z]]$ via $w \mapsto z/(z^2 + \eta)$, to obtain an equation of the form

$$A(t) = \frac{\eta Z(t)/(Z(t)^2 + \eta)^2 - Y(t)}{Z(t)^3/(Z(t)^2 + \eta)^2 - Y(t)} \Big|_{t=M_t}. \quad (5.60)$$

Note again that $\text{ord}(W(t)) = -\text{ord}(Z(t)) \leq \text{ord}(Y(t))$.

The symmetric difference case is more complicated, as now the anticanonical curve is ramified over the base of the ruling, with natural parametrizations (assuming characteristic 0)

$$(y, x, w) = (1/z^3, 1, 1/z^2) \quad \text{and} \quad (y, x, w) = (-1/z^3, 1, 1/z^2) \quad (5.61)$$

of the anticanonical curve $y^2 = w^3x$. Note that as before, we have $\text{ord}(W(t)) \leq \text{ord}(Y(t))$, and since the characteristic is 0 we may assume that $W(t) = t^a$. Let $Y(t) = \sum_{b \leq l} c_b t^b$, with $c_b \neq 0$. Then the eigenvalues of $Y(M_t)$ are the Puiseux series of the form

$$\sum_{b \leq l} c_b z^{-2l/a}, \quad (5.62)$$

one for each of the a roots $z^{1/a}$, and thus the eigenvalues of $A(z)$ are the different values of

$$\frac{1/z^3 - \sum_{b \leq l} c_b z^{-2l/a}}{-1/z^3 - \sum_{b \leq l} c_b z^{-2l/a}}. \quad (5.63)$$

Note that if a is odd, then the eigenvalues are all ± 1 at $z = \infty$, depending on the sign of $3 - 2b/a$.

For (1), the answer is in one sense trivial: given a (spectral) map from $\text{Spec}(k[[t]])$ to F_2 , we simply repeatedly blow up the image of the closed point until the closed point is no longer on the anticanonical curve. This is reasonably straightforward, but as stated tells us little about the inverse problem. After each blowup, we obtain a patch of X_m containing the image of the special fiber, and a morphism from that patch to X_{m-1} . Apart from the parametrized spectral curve itself, the other main piece of information on each patch is the equation of the anticanonical curve. A key observation is that after the first few blowups, there are at most two (local) components of the anticanonical curve, and those components (which may have multiplicity) meet transversely. In other words, after a few steps, we can coordinatize the resulting patch so that the anticanonical curve has the equation $u^{m_1}v^{m_2} = 0$, where the most recent exceptional curve is $u = 0$. (If $m_2 = 0$, there is some ambiguity in the coordinate v , but we always at least choose it so that the point being blown up is the origin.) Thus the question factors into understanding the first few steps and understanding this general scenario. Note that since we are assuming that we never blow up a point of an exceptional curve which is not an anticanonical component, we may assume that $m_1 > 0$.

Let $u(t), v(t) \in tk[[t]]$ be the parametrization of the spectral curve in this patch. Note that we may assume that $u(t)^{m_1}v(t)^{m_2}$ is not identically 0, since otherwise the spectral curve would not be transverse to the anticanonical curve. After blowing up the origin, there are three possibilities for the new affine patch. The new exceptional curve is $\hat{u} = 0$, and the other relevant component (if any) of the anticanonical curve is $\hat{v} = 0$.

- (a) If $\text{ord}(u) > \text{ord}(v)$, then the new patch has coordinates $\hat{u}(t) = v(t)$, $\hat{v}(t) = u(t)/v(t)$, with $\hat{m}_1 = m_1 + m_2 - 1$, $\hat{m}_2 = m_1$.
- (b) If $\text{ord}(u) < \text{ord}(v)$, then the new patch has coordinates $\hat{u}(t) = u(t)$, $\hat{v}(t) = v(t)/u(t)$, with $\hat{m}_1 = m_1 + m_2 - 1$, $\hat{m}_2 = m_2$.
- (c) If $\text{ord}(u) = \text{ord}(v)$, then the new patch has coordinates $\hat{u}(t) = u(t)$, $\hat{v}(t) = v(t)/u(t) - \alpha$, where $\alpha = \lim_{t \rightarrow 0} v(t)/u(t)$, $\hat{m}_1 = m_1 + m_2 - 1$, $\hat{m}_2 = 0$.

Note that if $m_2 = 0$, then (b) and (c) give the same combinatorial type, so we merge them into a case (bc). Geometrically, these cases correspond to the case that the spectral curve is tangent to $u = 0$, tangent to $v = 0$, or tangent to neither, respectively. Every time we hit case (c) or (bc) we pick up a parameter α , which is in k^* when $m_2 \neq 0$ and k when $m_2 = 0$.

If $m_1 = 1$, $m_2 = 0$, then the intersection with the anticanonical curve depends only on the order of vanishing $\text{ord}(u)$ (which by our genericity assumptions must be 1). Note here that the choice of point to blow up on the exceptional curve was made at the *previous* step, and thus there are no more parameters (since this blowup separates the spectral and anticanonical curves). We find the following by an easy induction.

- (1) If $m_1, m_2 > 0$, then the intersection of the image of the structure sheaf with C_α depends only on $u(t) + o(t^{m_1 \text{ord}(u) + (m_2 - 1) \text{ord}(v)})$ and $v(t) + o(t^{(m_1 - 1) \text{ord}(u) + m_2 \text{ord}(v)})$.
- (2) If $m_1 > 0$, $m_2 = 0$, then it depends only on $u(t) + o(t^{m_1 \text{ord}(u)})$ and $v(t) + o(t^{(m_1 - 1) \text{ord}(u)})$.

In each case, we either have $(m_1, m_2) = (1, 0)$ or have enough information to determine in which case (a-c) we are in, and in case (c) what the value of the parameter is. Moreover, in each case, we find that we know the new coordinates to at least the correct precision.

Note in particular that if $(m_1, m_2) = (1, 1)$, then the combinatorial type depends only on $\text{ord}(u)$ and $\text{ord}(v)$. Moreover, we see that the blowing up process is essentially just performing the

subtraction form of Euclid's algorithm for computing $\gcd(\text{ord}(u), \text{ord}(v))$, and the additional data determining the intersection is just the constant term of

$$u(t)^{\text{ord}(v)/\gcd(\text{ord}(u), \text{ord}(v))} v(t)^{-\text{ord}(u)/\gcd(\text{ord}(u), \text{ord}(v))}. \quad (5.64)$$

The nonreduced cases (i.e., $\max(m_1, m_2) > 1$) are more complicated to deal with, and we can only give a satisfying description in characteristic 0. The advantage of working in characteristic 0 is that we can take roots of power series. In particular, we know u to at least enough precision to know its leading term, and thus we may choose a new coordinate s such that $u(s) = s^{\text{ord}(u)}$, and find that we now need to specify only the coordinate $v(s)$, to precision $o(s^{(m_1-1)\text{ord}(u)+m_2(\text{ord}(v))})$. If $m_2 > 0$, we could instead reparametrize so that $v(s) = s^{\text{ord}(v)}$, so that the only data is $u(s) + o(s^{m_1\text{ord}(u)+(m_2-1)\text{ord}(v)})$. Note that knowing

$$v(s) + o(s^{(m_1-1)\text{ord}(u)+m_2(\text{ord}(v))}) \quad (5.65)$$

tells us

$$v(s)^{1/\text{ord}(v)} + o(s^{1+(m_1-1)\text{ord}(u)+(m_2-1)(\text{ord}(v))}), \quad (5.66)$$

and thus tells us the inverse function to the same precision. In other words, when both parametrizations are defined, the information we obtain is the same in either case.

To proceed further, we need to understand better how the two parametrizations are related. To begin with, let $H \subset \mathbb{N}$ be a submonoid (i.e., $0 \in H$ and H is closed under addition). A power series supported on H is then an element of $k[[t]]$ in which the coefficient of t^l is 0 unless $l \in H$.

Lemma 5.29. *For any submonoid $H \subset \mathbb{N}$, the power series supported on H form a ring, in which the series with nonzero constant terms are units. If $f \in k[[t]]$ is supported on H with $f(0) = 0$, then for any other element $g \in k[[t]]$, $g \circ f$ is supported on H .*

Proof. That the power series supported on H form a ring is trivial, and thus any polynomial in such a power series is also supported on H . Since the set of power series supported on H is closed under formal limits, the claim regarding composition follows. That the reciprocal of a series with nonzero constant term is still supported on H then follows by plugging in $f - f(0)$ into the appropriate geometric series. \square

More surprisingly, we have the following.

Lemma 5.30. *For any submonoid $H \subset \mathbb{N}$, the power series with $f(0) = 0$, $f'(0) \neq 0$ and $f(t)/t$ supported on H form a group under composition.*

Proof. Suppose g is such a series, with

$$g(t) = \sum_{n \in H} c_n t^{n+1}, \quad (5.67)$$

$c_0 \neq 0$. Then

$$g(f(t)) = \sum_{n \in H} c_n (f(t))^{n+1} = \sum_{n \in H} c_n t t^n (f(t)/t)^{n+1}. \quad (5.68)$$

Since $(f(t)/t)^{n+1}$ is supported on H and t^n is supported on H , we find that these series are indeed closed under composition.

It remains only to show that the compositional inverse of such a series is again of this form. Let $f(t)$ be such a series, and let $g(t)$ be its compositional inverse. Suppose $g(t)/t$ is not supported on

H , and let n be the minimal nonnegative integer not in H such that the coefficient of t^{n+1} in $g(t)$ is nonzero. We may thus write

$$g(t) = g_0(t) + c_n t^{n+1} + o(t^{n+1}), \quad (5.69)$$

where $g_0(t)/t$ is supported on H . We then have

$$t = g(f(t)) = g_0(f(t)) + c_n t^{n+1} (f(t)/t)^{n+1} + o(t^{n+1}), \quad (5.70)$$

so that

$$1 - \frac{g_0(f(t))}{t} = c_n t^n (f(t)/t)^{n+1} + o(t^n). \quad (5.71)$$

The left-hand side is supported on H , and thus the coefficient of t^n on the left vanishes, while on the right it equals

$$c_n f'(0)^{n+1} \neq 0, \quad (5.72)$$

giving the desired contradiction. \square

In particular, the monoid generated by the exponents of the nonzero terms of $s^{-\text{ord}(v)}v(s)$ in the $u(s) = s^{\text{ord}(u)}$ parametrization will be the same as that coming from the other parametrization, when both parametrizations are valid, and this monoid will be unchanged under blowup as long as we remain in scenarios (a) and (b). Unfortunately, scenario (c) can change the monoid in somewhat unpredictable ways. Luckily, we can enlarge the monoid without losing information about the blowup.

Call a subset $H \subset \mathbb{N}$ “arithmetic” if it contains 0 and for all $n \in H$, also contains $n + \gcd(\{l : l \in H \mid l \leq n\})$. Such a set is certainly a monoid, and we can still consider the arithmetic set generated by the exponents of nonzero terms of a power series. Note that an arithmetic set has a minimal set of generators: d_1 is the minimal nonzero element, d_2 the minimal element not a multiple of d_1 , d_3 the minimal element not a multiple of $\gcd(d_1, d_2)$, etc.

We associate an arithmetic set to our configuration $u(t), v(t)$ as follows. If $m_1, m_2 > 0$, then we change coordinates so that $u(s) = s^{\text{ord}(u)}$, and take the arithmetic set generated by $\gcd(\text{ord}(u), \text{ord}(v))$ and the exponents of the nonzero terms of $s^{-\text{ord}(v)}v(s)$. (Here, by convention, we suppose that all terms past the required precision are nonzero.) If $m_1 > 0, m_2 = 0$, then we take the same coordinate change, but now the arithmetic set is generated by $\text{ord}(u)$ and the exponents of nonzero terms of $v(s)$.

For $m_1, m_2 > 0$, we find that blowing up has no effect on the arithmetic set, while for $m_2 = 0$, scenario (bc) replaces the arithmetic set by $(H - \text{ord}(u)) \cap \mathbb{N}$, while scenario (a) replaces it by $H \cup \gcd(\text{ord}(u), \text{ord}(v))\mathbb{N}$. Moreover, for $m_2 = 0$, we can detect which of (a) or (bc) we are in knowing only $\text{ord}(u)$ and H ; scenario (a) is the case that H contains a nonzero element less than $\text{ord}(u)$.

Let us now see how this information gets used for the different types of equations. The q -difference cases are now the simplest, as the anticanonical curve can now already be written as $uv = 0$ in suitable (formal) coordinates. For the nonsymmetric q -difference case, the two coordinates are y and $w - y$, so the surface determines $a := \text{ord}(Y(t))$, $b := \text{ord}(W(t) - Y(t))$, and

$$\gamma := \lim_{t \rightarrow 0} \frac{Y(t)^b}{(W(t) - Y(t))^a}. \quad (5.73)$$

Moreover, the fact that we end up with multiplicity 1 implies that $\gcd(a, b) = 1$. If we write $Y(t) = \alpha t^a(1 + O(t))$, $W(t) - Y(t) = \beta t^b(1 + O(t))$, then we have three cases. If $a < b$, then

$W(t) = \alpha t^a(1 + O(t))$, so we may as well reparametrize to make $\alpha = 1$, $W(t) = t^a$, and thus the equation has the symbolic form

$$\frac{v(qz)}{v(z)} = \gamma^{1/a} z^{-1+b/a} (1 + O(z^{-1/a})), \quad (5.74)$$

where we translate Puiseux series to matrices as above. Similarly, in the case $a > b$, we have $W(t) = \beta t^b(1 + O(t))$, and again may take $\beta = 1$ to give

$$\frac{v(qz)}{v(z)} = \gamma^{1/b} z^{1-a/b} (1 + O(z^{-1/b})). \quad (5.75)$$

Finally, if $a = b = 1$, then $W(t) = (\alpha + \beta)(1 + O(t))$, and the equation is

$$\frac{v(qz)}{v(z)} = \gamma(1 + O(z^{-1})). \quad (5.76)$$

This is essentially the same as the formal classification of q -difference equations given in [48]. We obtain the same three cases in the symmetric q -difference case, as the needed reparametrizations do not affect the leading terms of the numerator and denominator of A .

For the differential case, we start with $(m_1, m_2) = (2, 0)$ with $u = y$, $v = w$. The above description of the spectral curve thus has the form

$$Y(t) = u(t) = t^a \quad (5.77)$$

$$W(t) = v(t) = \sum_{0 \leq l \leq a-b} c_l t^{l+b} + o(t^a), \quad (5.78)$$

with $c_b \neq 0$, where the corresponding arithmetic set is determined by the degrees where the gcds of the exponents up to that point (with a) drop. Note that the assumption that we meet the last -1 -curve with multiplicity 1 implies that the gcd actually reaches 1 before we hit $o(t^a)$. Moreover, our assumption of disjointness from $f - e_1$ implies $b \leq a$, and thus we could instead reparametrize to have

$$Y(t) = t^a \left(\sum_{0 \leq l \leq a-b} e_l t^l + o(t^{a-b}) \right), \quad W(t) = t^b, \quad (5.79)$$

where the e_l are determined from c_l and generate the same arithmetic set. This gives an equation of the form

$$\frac{zv'(z)}{v(z)} = \sum_{0 \leq l \leq a-b} f_l z^{(a-l)/b-1} + o(1), \quad (5.80)$$

where again the arithmetic set is the same. Note that we can record the combinatorial information in the equivalent form of specifying the leading exponent and each later exponent where the common denominator increases. Also, although the translation from the parameters of the surface to the coefficients f_l is fairly complicated, there is a one-to-one correspondence between those parameters and the potentially nonzero coefficients of the Puiseux series, such that each f_l depends only the parameters up to that point, and is degree 1 in the parameter corresponding to l , unless the parameter lies in k^* in which case it is proportional to an appropriate power of the parameter. In particular, if we are considering two irreducible singularities that share the first few steps of the blowup, then the corresponding Puiseux series must agree in precisely the first N terms (among those allowed to be nonzero), where N is the number of shared parameters.

For instance, suppose we are given an irreducible singularity of the form

$$\frac{zv'(z)}{v(z)} = f_0z^3 + f_6z^2 + f_{12}z + f_{14}z^{2/3} + f_{15}z^{1/2} + f_{16}z^{1/3} + f_{17}z^{1/6} + f_{18} + o(1). \quad (5.81)$$

Assuming f_0, f_{14}, f_{15} are nonzero, then the combinatorial data is determined by the sequence $(3, 2/3, 1/2)$, and the corresponding moduli space can be identified with $(k^*)^3 \times k^5$. The corresponding sequence of $(m_1, m_2)_{a/b/c}$ is:

$$\begin{aligned} & (2, 0)_a(1, 2)_b(2, 2)_b(\mathbf{3}, \mathbf{2})_c(\mathbf{4}, \mathbf{0})_{bc}(\mathbf{3}, \mathbf{0})_{bc} \\ & (2, 0)_a(1, 2)_b(\mathbf{2}, \mathbf{2})_c \\ & (3, 0)_a(\mathbf{2}, \mathbf{3})_c(\mathbf{4}, \mathbf{0})_{bc}(\mathbf{3}, \mathbf{0})_{bc}(\mathbf{2}, \mathbf{0})_{bc}(\mathbf{1}, \mathbf{0})_{bc}, \end{aligned}$$

where we have indicated the steps introducing parameters in bold. We should recall here that the subscript on a tuple $(m_1, m_2)_{a/b/c}$ determines the *next* point to be blown up. Note that the three parameters from k^* are

$$-\frac{1}{f_0}, \frac{f_0^6}{f_{14}}, \frac{f_{14}^9}{9f_0^{14}f_{15}^2}, \quad (5.82)$$

and the remaining parameters in k are even more complicated functions of the coefficients. The anticanonical curve on the resulting surface has the decomposition

$$\begin{aligned} & 2(s + f - e_1 - e_2 - e_3 - e_4) + (e_1 - e_2) + 2(e_2 - e_3) + 3(e_3 - e_4) + 4(e_4 - e_5) + 3(e_5 - e_6) \\ & + 2(e_6 - e_7 - e_8 - e_9) + (e_7 - e_8) + 2(e_8 - e_9) \\ & + 3(e_9 - e_{10} - e_{11}) + 2(e_{10} - e_{11}) + 4(e_{11} - e_{12}) + 3(e_{12} - e_{13}) + 2(e_{13} - e_{14}) + (e_{14} - e_{15}). \end{aligned}$$

Note that although setting a coefficient other than f_0, f_{14}, f_{15} to 0 has no effect on the combinatorics, setting one of the critical coefficients to 0 can make significant changes. For instance, the generic subcase with $f_{15} = 0$ has one fewer blowup, so this does not correspond to a degeneration of surfaces as considered above.

The one complication in the nonsymmetric difference case is that we do not start in a configuration $u^{m_1}v^{m_2} = 0$, since the two branches are tangent. We must blow up twice in order to achieve this, and this leads to several cases. Note first that blowing up a general point of e_1 yields an equation

$$\frac{v(z+1)}{v(z)} = 1 + \alpha/z + o(1/z), \quad (5.83)$$

$\alpha \neq 0$, which leaves the case that we blow up the triple intersection. This yields an anticanonical decomposition $(s + f - e_1 - e_2) + (s + f - e_1 - e_2) + (e_1) + 2(e_2)$, and there are four possibilities for the next blowup: a general point λ of e_2 , or the intersection with one of the other three components. The resulting patches, with the equation of the anticanonical curve and the map to the original coordinates are:

- (1) $u^2 = 0$, where $y = u^2(v + \lambda)$, $w = u$.
- (2) $u^2v = 0$, where $y = u^2v$, $w = u$.
- (3) $u^2v = 0$, where $y = u^2(v + 1)$, $w = u$.
- (4) $u^2v = 0$, where $y = u^2v$, $w = uv$.

In case (1), if we take $u(t) = t^a$, $v(t) = \sum_{b \leq l \leq a} c_l t^l + o(t^a)$, then the resulting equation has the symbolic form

$$\frac{f(z+1)}{f(z)} = \frac{\lambda + v(z^{-1/a})}{\lambda - 1 + v(z^{-1/a})} + o(z^{-1}) = \frac{\lambda}{\lambda - 1} + \sum_{b \leq l \leq a} d_l z^{-l/a} + o(z^{-1}), \quad (5.84)$$

where $d_b \neq 0$ and the common denominator increases at the same places as $v(z^{-1/a})$. In case (2), we now know $v(t)$ to relative precision $o(t^a)$, and now the equation is

$$\frac{f(z+1)}{f(z)} = \frac{v(z^{-1/a})}{v(z^{-1/a}) - 1} (1 + o(z^{-1})) = \sum_{b \leq l \leq a+b} d_l z^{-l/a} + o(z^{-b/a-1}), \quad (5.85)$$

where again the common denominator increases at the same points as $v(z^{-1/a})$. Case (3) is analogous, and simply gives the dual equation. Finally, for case (4), we want to parametrize so that $u(t)v(t) = t^a$, and have $v(t) = \sum_{b \leq l \leq a} c_l t^l + o(t^a)$ with $c_b \neq 0$, giving the equation

$$\frac{f(z+1)}{f(z)} = \frac{1}{1 - v(z^{-1/a})} = 1 + \sum_{b \leq l \leq a} d_l z^{-l/a} + o(z^{-1}), \quad (5.86)$$

now with $a > 1$, since e_3 is a multiple component of the fiber in this case. Again, these agree with the formal classification of difference equations [55, 40], apart from the description of the combinatorial data.

The symmetric difference case is, naturally, even more complicated, as now we may need to blow up three times to have just a pair of transverse branches to consider. The simplest case is that we blow up a general point of e_1 , which gives an equation of symbolic form

$$\frac{f(z+1)}{f(z)} = \exp(\alpha/z + O(1/z^3)). \quad (5.87)$$

(Here we ignore the effect that the shift has on the symmetry condition.) Similarly, blowing up the point of tangency then a general point of e_2 gives an equation of symbolic form

$$\frac{f(z+1)}{f(z)} = -\exp(\alpha/z + O(1/z^3)). \quad (5.88)$$

Otherwise, we are blowing up the triple intersection, and again have four cases.

- (1) $u^2 = 0$, where $y = u^3(v + \lambda)^2$, $w = u^2(v + \lambda)$.
- (2) $u^2 v = 0$, where $y = u^3 v^2$, $w = u^2 v$.
- (3) $u^2 v = 0$, where $y = u^3(v + 1)^2$, $w = u^2(v + 1)$.
- (4) $u^2 v = 0$, where $y = u^3 v$, $w = u^2 v$.

In case (1), we reparametrize so that $u(t)^2(\lambda + v(t)) = t^{2a}$, corresponding to $t = z^{-1/a}$, giving symbolic equations of the form

$$\frac{f(z+1)}{f(z)} = \frac{1 - \sqrt{\lambda}}{1 + \sqrt{\lambda}} \exp\left(\sum_{1 \leq l \leq a} c_l z^{-l/a} + o(1/z)\right). \quad (5.89)$$

Of course, the symmetry means that these equations come in pairs, corresponding to the fact that the full equation has even order. Case (3) is the next simplest, as we may still reparametrize so that $u(t)^2(1+v(t)) = t^{2a}$, giving again pairs of symbolic forms typified by

$$\frac{f(z+1)}{f(z)} = \frac{\sqrt{1+v(z^{-1/a})} - 1}{\sqrt{1+v(z^{-1/a})} + 1} = \sum_{b \leq l \leq a+b} c_l z^{-l/a} + o(z^{-1-b/a}) \quad (5.90)$$

For (2) and (4), the natural parametrization is $z^{-2} = u(t)^2 v(t) = t^{2a+b}$, where u and v have order a and b respectively, and are known to relative precision $o(t^a) = o(z^{-2a/(2a+b)})$. Case (4) then gives an equation of the form

$$\frac{f(z+1)}{f(z)} = \exp\left(\sum_{0 \leq l \leq a} c_l z^{-(b+2l)/(2a+b)} + o(1/z)\right). \quad (5.91)$$

If b is odd, this equation satisfies the symmetry condition, while if b is even, then the overall order is even and the symbolic equations come in pairs.

It is worth noting in each case how the corresponding canonical isomonodromy transformations behave. Note that when performing the isomonodromy transformation corresponding to twisting by e_l , the effect is to perform the d -th iterate of the twist by e_m , where d is the multiplicity of e_m in the pullback of e_l . We thus need only consider the twist by e_m . The result is then to replace the sheaf by the sheaf of functions vanishing at that point, and is thus easily seen to correspond to gauging by M_t (which we may view symbolically as the appropriate power of z). In the q -difference cases, gauging by $z^{-1/a} + o(z^{-1/a})$ multiplies the leading coefficient by $q^{-1/a}$; since the true parameter is the a -th power of the leading coefficient, this shifts the parameter by q as expected. In the nonsymmetric ordinary difference case, gauging by $z^{-1/a} + o(z^{-1/a})$ multiplies A by $1 - 1/az + o(1/z)$, and similarly for the symmetric case. And of course in the differential case gauging by $z^{-1/a}$ adds $-1/a + o(1)$ to zf'/f .

In the differential case, we of course also have continuous isomonodromy transformations, which in the Puiseux form correspond to equations

$$\frac{f_u}{f} = \frac{a}{l+a} z^{l/a+1} \quad (5.92)$$

giving

$$\frac{d}{du} A = z^{l/a}. \quad (5.93)$$

In particular, for a minimal type with l parameters, we obtain $l-1$ such deformations. (There is also a deformation changing the location of the singularity.)

In the nonsymmetric difference case, if we write the symbolic equation as

$$\frac{f(z+1)}{f(z)} = z^{b/a} \exp(g(z^{-1/a})) \quad (5.94)$$

then the equation

$$f_u(z) = \frac{a}{a+l} z^{1+l/a} f(z) \quad (5.95)$$

gives

$$\frac{d}{du} g(z^{-1/a}) = z^{l/a} (1 + o(1/z)) \quad (5.96)$$

again giving continuous deformations for every parameter but the last. The symmetric difference case is analogous; the only change is that the continuous equation needs to preserve the symmetry, but this is easy to arrange.

If we take into account the symmetries (e.g., the PGL_2 symmetry in the differential case), we find in general that the net number of *local* continuous isomonodromy deformations is 0 if the anticanonical curve is reduced, and otherwise can be expressed as $1 - C_\alpha \cdot (C_\alpha - C_\alpha^{\mathrm{red}})$. This follows by an easy induction: blowing up a point of the smooth locus of C_α subtracts e_m from C_α^{red} , and similarly for blowing up a point on two components, while blowing up a point on a single component with nontrivial multiplicity leaves C_α^{red} alone. One thus reduces to the cases in which C_α first becomes nonreduced: $C_\alpha = 2(s + f)$, which has -3 deformations due to the symmetries, $C_\alpha = (s + f - e_1 - e_2) + (s + f - e_1 - e_2) + (e_1 - e_2) + 2(e_2)$, with no continuous deformations, and $C_\alpha = (2s + 2f - 2e_1 - e_2 - e_3) + (e_1 - e_2 - e_3) + (e_2 - e_3) + 2(e_3)$, likewise.

The fact that the result depends only on the geometry suggests that there should be a geometric explanation of these continuous isomonodromy deformations, just as the discrete isomonodromy deformations come from twists by line bundles. Of course, this assumes that the local deformations can in fact be glued together to form global deformations, though the results of [39] suggest that this is indeed the case.

6 Moduli of sheaves on surfaces

Let (X, C_α) be an anticanonical rational surface. Say a coherent sheaf on X has *integral support* if its 0-th Fitting scheme is an integral curve on X , and it contains no 0-dimensional subsheaf.

Theorem 6.1. *Let (X, C_α) be an anticanonical rational surface over an algebraically closed field of characteristic p , let D be a divisor class with generic representative an integral curve disjoint from C_α , and let r be the largest integer such that $D \in r \mathrm{Pic}(X)$. Then the moduli problem of classifying sheaves M on X with integral support, $c_1(M) = D$, $\chi(M) = x$, and $M|_{C_\alpha} = 0$ is represented by a quasiprojective variety $\mathrm{Irr}_X(D, x)$ of dimension $D^2 + 2$, with a symplectic structure induced by any choice of nonzero holomorphic differential on C_α . Moreover, $\mathrm{Irr}_X(D, x)$ is unirational if the generic representative of D has no cusp, separably unirational if $p = 0$ or $\gcd(x, r, p) = 1$, and rational if $x \bmod r \in \{1, r - 1\}$. Finally, if $\gcd(x, r) = 1$, then there exists a universal sheaf over $\mathrm{Irr}_X(D, x)$.*

Proof. Quasiprojectivity follows from the standard GIT construction: for any choice of stability condition, a sheaf with integral support is stable. The symplectic structure follows from the fact that sheaves with integral support are simple (have no nonscalar endomorphisms) together with the results of [42] (see also [22, 9]). This requires a choice of Poisson structure on X , or equivalently a choice of nonzero holomorphic differential on C_α ; the symplectic structure on the moduli space scales linearly with the choice of differential.

Now, the typical sheaf in the moduli space corresponds to a pair (C, M) where C is an integral curve of class D (and disjoint from C_α) and M is a torsion-free sheaf on C . If $g = D^2/2 + 1$, then $\Gamma(\mathcal{L}(D))$ has dimension $g + 1$, and thus the integral curves in the linear system form an open subset of a \mathbb{P}^g . We also compute that C has arithmetic genus g ; the fiber over the point corresponding to C is a compactification of $\mathrm{Pic}^{x+g-1}(C)$, so has dimension g . (As we might expect from a natural fibration of a symplectic scheme by half-dimensional subschemes, this is a Lagrangian fibration.)

If $x = 1$, then $\deg(M) = g$, and thus the generic such sheaf has a unique global section. The quotient by the corresponding trivial subsheaf is supported on g points, thus giving a birational correspondence with the punctual Hilbert scheme $X^{[g]}$. Since symmetric powers of rational surfaces are rational varieties, it follows that $\mathrm{Irr}_X(D, 1)$ is rational.

More generally, since D/r is a primitive element of the Picard lattice of X , there exists a divisor D' such that $D \cdot D' = r$. In particular, we can twist by powers of this divisor to obtain isomorphism $\mathcal{Irr}_X(D, x) \cong \mathcal{Irr}_X(D, r + x)$ for any x . Similarly, the duality morphism $M \mapsto \mathcal{H}om(M, \omega_C)$ on invertible sheaves can be defined globally (and extended to torsion-free sheaves) by $M \mapsto \mathcal{E}xt^1(M, \omega_X)$, and gives an isomorphism $\mathcal{Irr}_X(D, x) \cong \mathcal{Irr}_X(D, 2g - 2 - x)$. Since $2g - 2 = D^2$ is a multiple of r^2 , the rationality claim follows.

For unirationality, note that since $2g - 2$ is a multiple of r^2 , $g - 1$ is a multiple of r , and thus $\mathcal{Irr}_X(D, 2 - g)$ (classifying sheaves of degree 1) is rational. Since the generic sheaf is an *invertible* sheaf on the generic curve, we can take its d -th power and thus obtain a rational map $\mathcal{Irr}_X(D, 2 - g) \rightarrow \mathcal{Irr}_X(D, d + 1 - g)$. Since the generic curve C_{gen} is integral, this map is dominant unless C_{gen} is cuspidal and d is 0 in k . This implies unirationality in the noncuspidal case for any $d \neq 0$; the case $d = 0$ reduces to the case $d = g - 1$ by twisting. When $\gcd(d, p) = 1$, the multiplication by d map is separable, and again we may feel free to add multiples of r to make this happen.

Finally, we note that the obstruction to the existence of a universal on $\mathcal{Irr}_X(D)$ is given by a class in $H^2(\mathcal{Irr}_X(D), \mathbb{G}_m)$, which can be constructed as follows. There certainly exists a universal sheaf étale-locally, and we may twist by a line bundle to ensure that this universal sheaf is acyclic; i.e., that $x \geq g$. Then, since the fibers of the universal sheaf are simple, the endomorphism ring of the direct image descends to an Azumaya algebra on $\mathcal{Irr}_X(D)$, the class of which is the desired obstruction. This Azumaya algebra has degree x , and thus the obstruction has order dividing x . Since this is true for any twist of sufficiently large degree, we also find that the order of the obstruction divides $x + r$, and thus that it divides $\gcd(x, r) = 1$. In particular, if x and r are relatively prime, there is no obstruction. \square

Remark 1. The generically cuspidal case can of course only occur in finite characteristic, but can certainly occur there, say if D is the class of a fiber in a rational quasi-elliptic surface.

Remark 2. Often in the literature, one restricts one's attention to the subscheme where C is not just integral but smooth, making the fibers of the Lagrangian fibration abelian varieties. Of course, this is problematical in finite characteristic, where there may not be any smooth curves in the linear system. In addition, since singularity is a codimension 1 condition, this removes an entire hypersurface from the moduli space, based on a condition which is rather unnatural from the difference equation perspective. (Indeed, as we mentioned, difference equations correspond most naturally to sheaves on noncommutative surfaces, and there the notion of support fails altogether. In contrast, the failure of integrality corresponds to reducibility of the equation in a suitable sense.) For instance, in the generic 2-dimensional case, both the surface and the moduli space are elliptic surfaces, and there are 12 fibers where the support is singular. Similarly, there are 12 points of the moduli space where the sheaf is not invertible on its support, again an odd condition in terms of difference equations.

Remark 3. It seems likely that the condition $\gcd(x, r) = 1$ for the existence of a universal sheaf is necessary. Some condition is needed, see the remark following Proposition 6.3 below.

The rational case $x = 1$ is particularly nice for another reason: although the definition of stability generally requires the choice of an ample bundle, it turns out that when $\chi = 1$, this choice is irrelevant. One finds in this case M is stable iff any nonzero quotient of M has positive Euler characteristic (and there are no strictly semistable sheaves). We thus find that $\mathcal{Irr}_X(D, 1)$ extends naturally to a projective moduli space. This space is no longer symplectic, but since every sheaf in the space is stable, thus simple, it still inherits a Poisson structure. This Poisson variety has

smooth symplectic leaves determined by the quasi-isomorphism class of the complex $M \otimes^{\mathbf{L}} \mathcal{O}_{C_\alpha}$, see [42]. In particular, the open subvariety where $M|_{C_\alpha} = 0$ is still smooth and symplectic.

For our purposes, the most natural case is $x = D \cdot f$. Indeed, the sheaf corresponding to a difference equation comes from a sheaf on a Hirzebruch surface with presentation

$$0 \rightarrow \rho^* V \otimes \mathcal{O}_\rho(-1) \rightarrow \mathcal{O}_X^n \rightarrow M_0 \rightarrow 0. \quad (6.1)$$

If we twist by $-f$, then both sheaves in the resolution have vanishing cohomology, and thus $H^*(M_0 \otimes \mathcal{L}(-f)) = 0$; conversely, by Lemma 2.7, any sheaf with $H^*(M_0 \otimes \mathcal{L}(-f)) = 0$ at the least has a canonical subsheaf with a presentation of the above form. (In Section 2, we imposed the additional open conditions $\text{Hom}(M, \mathcal{O}_f(-1)) = \text{Hom}(\mathcal{O}_f(-1), M) = 0$ for all f ; ignoring those conditions gives us a natural partial compactification.)

Of course, we do not have a sheaf on a Hirzebruch surface, but rather a sheaf on some blowup of the Hirzebruch surface. However, we have the following fact, by the same spectral sequence argument as Lemma 2.7.

Lemma 6.2. *Let $\pi : X \rightarrow X_0$ be a birational morphism of smooth projective surfaces, and let M be a 1-dimensional sheaf on X . Then $H^0(M) = H^1(M) = 0$ iff M is π_* -acyclic and $H^0(\pi_* M) = H^1(\pi_* M) = 0$.*

In other words, a sheaf M on X induces a difference equation (up to twisting by $-f$) iff $H^0(M) = H^1(M) = 0$. (Again, it could fail to be the natural sheaf associated to a difference equation, but this can be avoided by imposing the additional conditions $\text{Hom}(M, \mathcal{O}_g(-1)) = \text{Hom}(\mathcal{O}_g(-1), M) = 0$ for any smooth rational curve g contained in a fiber.) We are thus led to consider the space $\text{Irr}_X(D, 0)$. Once again, the stability condition turns out to be independent of the choice of ample bundle: a 1-dimensional sheaf M on X with $\chi(M) = 0$ is stable iff any proper nontrivial subsheaf has negative Euler characteristic, and similarly for semistability. Since we need semistable sheaves, we do not immediately inherit a Poisson structure, although this will certainly exist on the complement of the semistable locus.

It remains only to consider the condition $H^0(M) = H^1(M) = 0$. By Lemma 5.3, this is the complement of a codimension 1 condition on any family of 1-dimensional sheaves, cutting out a Cartier divisor. Of course, this is only well-defined outside the semistable locus, but the generic sheaf is integral, and thus stable, so we still obtain a well-defined divisor on the projective moduli space. On the integral locus, this divisor is just the canonical theta divisor in the relative Pic^{g-1} , while in general, it is the zero locus of a canonical global section of a canonical line bundle $\det \mathbf{R}\Gamma(M)^{-1}$.

We can obtain a whole family of such divisors by noting that for any vector $v \in D^\perp$, we can twist M by $\mathcal{L}(v)$ without affecting the Euler characteristic, thus obtaining rational automorphisms of the projective moduli space (these are only rational, since twisting can affect stability; but this is an automorphism on the integral locus). In particular, we obtain in this way a canonical global section of $\det \mathbf{R}\Gamma(M \otimes \mathcal{L}(v))^{-1}$, which we call a “tau function” by analogy with [1]. It is of course a misnomer to call it a function (just like a theta function is not an algebraic function), but it is at the very least a convenient way of describing divisors on the moduli space. (Similarly, the divisor on the moduli stack of surfaces where a given divisor class is a -2 -curve can also be viewed as a tau function, as can theta functions themselves.)

The moduli space is 0-dimensional when D is a -2 -curve, and in that case, M is uniquely determined by its Chern classes and the constraint that it be disjoint from C_α ; moreover, its image on X_0 is similarly determined by its bidegree and its intersection with C_α ([42, Prop. 8.9]). Since the moduli spaces are symplectic, the next interesting case is the 2-dimensional case $D^2 = D \cdot K_X = 0$.

The only such divisors in the fundamental chamber are the classes $-rK_{X_8}$ for integer r , such that $\omega_{X_8}|_{C_\alpha}$ has exact order r in $\text{Pic}(C_\alpha)$. Now, in that case, X_8 is itself an quasi-elliptic surface, and D is the class of a fiber. The corresponding moduli spaces are just the relative Picard varieties of this quasi-elliptic surface.

Proposition 6.3. *Let $\psi : X \rightarrow \mathbb{P}^1$ be a relatively minimal rational quasi-elliptic surface. Then for any integer x , the relative Pic^x of X over \mathbb{P}^1 is a rational surface.*

Proof. This is essentially a result of [10, Prop. 5.6.1]. To be precise, that Proposition shows that a relatively minimal quasi-elliptic surface is rational iff it has at most one multiple fiber (and that “tame”) and its relative Jacobian is rational. Since the relative Jacobian of the relative Pic^x is isomorphic to the original relative Jacobian, and the relative Pic^x cannot turn non-multiple fibers into multiple fibers (or tame multiple fibers into wild multiple fibers), the claim follows. \square

Remark. It is worth noting that although the moduli space here is rational, there is in general no universal sheaf, even if we restrict to an open subset (or the function field) of the moduli space. In particular, when $x = 0$, it follows from Bhatt’s appendix to [31] that the obstruction can be identified with the class of X as a torsor over the relative Jacobian. In particular, we find in that case that the obstruction is nontrivial whenever $r > 1$.

Since our surfaces are anticanonical, it makes sense to ask which rational surface one obtains in this way. That is, if we start with a blowdown structure on X and a section of the anticanonical linear system, is there a natural way to choose a blowdown structure on the (minimal proper regular model of the) relative Pic^x such that we can compute the new anticanonical curve and the new morphism $\mathbb{Z}^{10} \rightarrow \text{Pic}(C'_\alpha)$? Note that on a (quasi-)elliptic rational surface, the line bundle $\omega_X|_{C_\alpha}$ has finite order (say r), and thus determines a subgroup of degree $\gcd(x, r)$. Since the corresponding bundle for the relative Pic^x has order $r/\gcd(x, r)$, the resulting surface should depend only on the composition $\mathbb{Z}^{10} \rightarrow \text{Pic}(C_\alpha)$ with the quotient by this subgroup. We will show this in the case $\gcd(x, r) = r$, and give an explicit description of the surface, in the following section.

Past the 2-dimensional cases, it no longer makes much sense to ask which variety we obtain (since birational geometry is extremely complicated, even for 4-folds). It is fairly straightforward to write down divisor classes giving such moduli spaces, however; for instance the 4-dimensional moduli spaces correspond (in the fundamental chamber of an even blowdown structure) to one of

$$2s + 3f - \sum_{1 \leq i \leq 10} e_i. \quad \text{or} \quad 4s + 4f - 2 \sum_{1 \leq i \leq 7} e_i - e_8 - e_9$$

The former is always generically integral (assuming of course that the corresponding line bundle is trivial on C_α), while the latter is generically integral unless $e_8 - e_9$ is a -2 -curve. Both of these cases extend to a sequence of moduli problems of dimension $2g$ for arbitrary $g > 1$. The first case extends to the general problem of classifying second-order problems with simple singularities (related to generalized Garnier-type systems); for the second case, see Section 10 below.

In general, for any dimension bigger than 2, there are only finitely many possibilities for the representative of D in the fundamental chamber. Indeed, we may write any divisor D in the fundamental chamber in the form

$$D = D_7 + c(2s + 2f - e_1 - \cdots - e_8) - \sum_i \lambda_i e_{8+i}, \quad (6.2)$$

where $D_7 \in \langle s, f, e_1, \dots, e_7 \rangle$ is in the fundamental chamber and λ is a partition with all parts at most c . The constraint $D \cdot C_\alpha = 0$ becomes

$$|\lambda| = D_7 \cdot C_\alpha \quad (6.3)$$

and we have

$$D^2 = D_7^2 + 2cD_7 \cdot C_\alpha - \sum_{1 \leq i} \lambda_i^2 \geq D_7^2 + cD_7 \cdot C_\alpha. \quad (6.4)$$

Since every generator of the fundamental chamber for $m = 7$ has positive intersection with C_α , there are only finitely many pairs (D_7, c) such that $D_7 \neq 0$ and $D_7^2 + cD_7 \cdot C_\alpha \leq d - 2$, where d is our desired value for $D^2 + 2$. (The pairs with $D_7 = 0$ correspond to the $D^2 = 0$ case.)

7 Moduli of sheaves on rational (quasi-)elliptic surfaces

Let (X, Γ, C_α) be an anticanonical rational surface with blowdown structure Γ , and suppose that X is (quasi-)elliptic, so that the divisor rC_α is the class of a fiber of a genus 1 pencil for some positive integer r . Since $-rK$ is the class of a pencil, we see that the line bundle $\omega_X^r|_{C_\alpha}$ is trivial. Moreover, if $\omega_X|_{C_\alpha}$ had order s strictly dividing r , then sC_α would already have multiple sections, contradicting the assumption on rC_α . We thus find that in this scenario, $\omega_X|_{C_\alpha}$ has exact order r . Per Proposition 5.14, such a torsion bundle induces an automorphism of C_α of order r , and we can quotient by this automorphism to obtain a new curve C'_α . The pair (X, Γ) is determined from the map $\Lambda_{10} \rightarrow \text{Pic}(C_\alpha)$; if we compose with the degree-preserving map $\text{Pic}(C_\alpha) \rightarrow \text{Pic}(C'_\alpha)$, we obtain a new map $\Lambda_{10} \rightarrow \text{Pic}(C'_\alpha)$. This new map has the same combinatorial structure as the original map (twisting by $\omega_X|_{C_\alpha}$ preserves degrees, so the automorphism preserves components), and thus itself arises from a unique triple (X', Γ', C'_α) .

Theorem 7.1. *Suppose C_α is reduced. Then the surface X' constructed in this way is the minimal proper regular model of the relative Pic^r of X , in such a way that the fiber corresponding to rC_α is C'_α .*

Proof. Assume for the moment that the sublattice of Λ_{E_8} corresponding to C_α is saturated; this excludes only two cases, namely $\Lambda_{A_7} \subset \Lambda_{E_7} \subset \Lambda_{E_8}$ and $\Lambda_{A_8} \subset \Lambda_{E_8}$, which we will discuss below. (The remaining unsaturated sublattice Λ_{D_8} corresponds to a nonreduced C_α .) Together with the hypothesis that C_α is reduced, this is equivalent to assuming that $\text{Pic}(X)$ is generated by the -1 -classes meeting each component of C_α positively. In particular, this ensures that the homomorphism $\text{Pic}(X) \rightarrow \text{Pic}(C_\alpha)$ is determined by its restriction to such classes. A -1 -class is always uniquely effective, the transversality condition implies that the corresponding curve meets C_α in a single smooth point, and that point in turn determines the image in $\text{Pic}^1(C_\alpha)$. In particular, we can reconstruct X from C_α and the configuration of points in which -1 -classes meet C_α , and similarly for X' . As a result, to prove the theorem, we will simply need to show that the minimal proper regular model of the relative Pic^r has the correct special fiber, and has the relevant -1 -classes, meeting C'_α in the correct points.

Rather than study the minimal proper regular model directly, we instead consider the corresponding moduli space of semistable sheaves with first Chern class $c_1(M) = -rK$ and Euler characteristic $\chi(M) = r$. Unlike the cases $\chi \in \{-1, 0, 1\}$ discussed earlier, in this case the stability condition depends nontrivially on the choice of ample divisor $\mathcal{O}_X(1)$. There are only finitely many divisors with $D, -rK - D$ effective (i.e., subdivisors of fibers); we may thus choose the ample divisor in such a way that for any such divisor, either $D \in \mathbb{Z}K$ or

$$\frac{D \cdot \mathcal{O}_X(1)}{K \cdot \mathcal{O}_X(1)} \notin \mathbb{Z}. \quad (7.1)$$

This ensures that any semistable sheaf not supported on the special fiber will be stable; the various inequalities are forced to be strict by integrality. This will also force any semistable sheaf supported

on the special fiber to be S -equivalent to a sum of stable sheaves supported on C_α , see below. (We should also note that for any fixed $r \geq 1$, that there are only finitely many divisor classes with both D and $-rK - D$ effective on *some* triple (X, Γ, C_α) , and could thus in principle choose the ample divisor in a uniform way over any family of elliptic surfaces.)

Since stable sheaves are simple (and stability is an open condition), we find that the moduli space has an open subset which agrees with an open subset of the moduli space of simple sheaves. Any sheaf in the open subset has support disjoint from C_α , and thus that open subset has a natural symplectic structure. In particular, we find that this open subset is smooth. It is also minimal, in that it cannot contain any -1 -curve of a smooth compactification (e.g., the desired minimal proper regular model). (More precisely, one can define intersections of line bundles with projective curves in quasiprojective surfaces; that any quasiprojective symplectic surface is minimal follows by noting that $0 = K \cdot e = -1$ for any curve e which can be blown down.) And, of course, it has a natural fibration over an affine line (induced by that of the complement of C_α in X) such that the smooth locus of the generic fiber is Pic^r of the corresponding fiber of X . In other words, this open subset of the moduli space is precisely the minimal proper regular model of the relative Pic^r of $X \setminus C_\alpha$. It is thus natural to conjecture that the Zariski closure of this open subset is the full minimal proper regular model. We call this Zariski closure the *main component* of the semistable moduli space (in fact, it is typically the smallest, but most interesting, component).

It will thus be necessary to understand the remainder of the moduli space. A key observation is that if M is semistable and supported on rC_α , then M is S -equivalent to a sheaf scheme-theoretically supported on C_α . Indeed, $\omega_X|_{C_\alpha}$ is in the identity component of $\text{Pic}(C_\alpha)$, and thus twisting by ω_X preserves semistability. If for some $l > 0$, M is supported on $(l+1)C_\alpha$ but not on lC_α , then we have a nonzero morphism $M \otimes \omega_X \rightarrow M$ between semistable sheaves of the same slope, and thus M is S -equivalent to the sum of the image and the cokernel of this morphism. This makes M S -equivalent to a sheaf supported on lC_α , and we may proceed by induction in l .

Lemma 7.2. *Let C_α be an anticanonical curve on a rational surface X . If M is a semistable sheaf supported on C_α with $c_1(M) = rC_\alpha$, $\chi(M) = r$, then M is S -equivalent to a sum of stable sheaves with $c_1(M) = C_\alpha$, $\chi(M) = 1$.*

Proof. We first claim that the slope 0 sheaf \mathcal{O}_{C_α} is stable. We need to show that any quotient sheaf has positive Euler characteristic, and can easily reduce to the case of a torsion-free quotient, i.e., \mathcal{O}_D for some curve $D \subset C_\alpha$. But by Lemma 5.13, we have $\chi(\mathcal{O}_D) = h^0(\mathcal{O}_D) > 0$ as required. It follows immediately that the ideal sheaf of a point is stable, since all of its subsheaves are subsheaves of \mathcal{O}_{C_α} , so have negative Euler characteristic. Then, by duality on X , we obtain stability of the sheaf

$$\mathcal{O}_{C_\alpha}(p) := \mathcal{E}xt_X^1(\mathcal{I}_p, \omega_X), \quad (7.2)$$

the unique nontrivial extension of \mathcal{O}_p by \mathcal{O}_{C_α} .

It will thus suffice to show that M is S -equivalent to a sum of sheaves of the form $\mathcal{O}_{C_\alpha}(p)$. In fact, it will suffice to construct a nonzero homomorphism from M to some $\mathcal{O}_{C_\alpha}(p)$: since $\mathcal{O}_{C_\alpha}(p)$ is stable of the same slope as M (true regardless of the choice of ample divisor!), such a morphism is necessarily surjective. The kernel of the surjection will then remain semistable of the same slope, and we may proceed by induction. The same argument shows that $\text{Hom}(M, \mathcal{O}_{C_\alpha}) = 0$ (since \mathcal{O}_{C_α} is stable of smaller slope than M), and thus by duality (note that C_α is Gorenstein with trivial dualizing sheaf), $H^1(M) = 0$.

For any point $p \in C_\alpha$, consider the short exact sequence

$$0 \rightarrow M'_p \rightarrow M \rightarrow M \otimes \mathcal{O}_p \rightarrow 0. \quad (7.3)$$

If the map $H^0(M) \rightarrow H^0(M \otimes \mathcal{O}_p)$ fails to be surjective, or, equivalently,

$$\mathrm{Ext}^1(M \otimes \mathcal{O}_p, \mathcal{O}_{C_\alpha}) \rightarrow \mathrm{Ext}^1(M, \mathcal{O}_{C_\alpha}) \quad (7.4)$$

fails to be injective, then $\mathrm{Hom}(M'_p, \mathcal{O}_{C_\alpha}) \neq 0$, and any such morphism induces a nontrivial extension of $M \otimes \mathcal{O}_p$ by \mathcal{O}_{C_α} together with a nonzero morphism from M to this extension. (Indeed, considerations of Hilbert polynomials show that the image of this morphism has first Chern class C_α .) Since $\dim \mathrm{Ext}^1(\mathcal{O}_p, \mathcal{O}_{C_\alpha}) = 1$, this extension has the form $\mathcal{O}_p^n \oplus \mathcal{O}_{C_\alpha}(p)$, and thus M has a nonzero morphism to $\mathcal{O}_{C_\alpha}(p)$.

If the map $H^0(M) \rightarrow H^0(M \otimes \mathcal{O}_p)$ is always surjective, then the map $H^0(M) \otimes \mathcal{O}_{C_\alpha} \rightarrow M$ is surjective on fibers, and thus surjective. But since both sheaves have the same first Chern class and M has the larger Euler characteristic, this is impossible! \square

Remark 1. Note that one can reconstruct p from the sheaf $\mathcal{O}_{C_\alpha}(p)$, since the latter has a unique global section. It follows that there are at most r distinct points p_i admitting morphisms $M \rightarrow \mathcal{O}_{C_\alpha}(p_i)$. Since the cokernel of the natural morphism $H^0(M) \otimes \mathcal{O}_{C_\alpha} \rightarrow M$ is supported (set-theoretically) on those points, we conclude that the natural morphism is injective, and the cokernel is a 0-dimensional sheaf of degree r , from which we can read off the S -equivalence class of M .

Remark 2. If we replace semistability by the weaker condition that any nonzero quotient of M has positive Euler characteristic, we may still conclude by the same argument that M has a surjective morphism to some sheaf of the form $\mathcal{O}_{C_\alpha}(p)$. If we further replace $\chi(M) = r$ by $\chi(M) < r$, then $H^0(M) \rightarrow H^0(M \otimes \mathcal{O}_p)$ can never be surjective, so that M has a surjective morphism to *every* sheaf of the form $\mathcal{O}_{C_\alpha}(p)$.

We thus conclude that the portion of the moduli space classifying sheaves supported on the special fiber consists (up to S -equivalence) of sums of sheaves $\mathcal{O}_{C_\alpha}(p)$, and need to know which of these sheaves lie on the main component. The key additional constraint comes from the observation that if M is supported on $X \setminus C_\alpha$, then $M \otimes \omega_X \cong M$. Twisting by ω_X induces an automorphism of the full semistable moduli space, and it follows that this automorphism must act trivially on the main component. In other words, if M is S -equivalent to

$$\bigoplus_{1 \leq i \leq r} \mathcal{O}_{C_\alpha}(p_i), \quad (7.5)$$

the multiset of points p_i must be permuted by the action of ω_X . Since this action is free of order r on the smooth locus, we find that the S -equivalence classes fixed by the automorphism consist of sheaves

$$\bigoplus_{1 \leq k \leq r} \mathcal{O}_{C_\alpha}(p) \otimes \omega_X^k \quad (7.6)$$

with p in the smooth locus, together with sums

$$\bigoplus_{1 \leq i \leq r} \mathcal{O}_{C_\alpha}(p_i) \quad (7.7)$$

in which each p_i is a singular point of C_α . In our case, since C_α is reduced, the latter gives only finitely many points. The first family of sheaves is manifestly classified by the smooth locus C'_α , with the closure of C'_α containing in addition only the sheaves $\mathcal{O}_{C_\alpha}(p)^r$ with p singular. (There could in principle be isolated additional points in the main component, but we will see below that this cannot happen.)

The main difficulty at this point is that it is very difficult to determine tangent spaces to GIT quotients at semistable points (especially so in our case, since we only want the tangent vectors coming from a particular component). To get around this, we will consider one more moduli space.

The condition that a simple sheaf is invertible on its support is open (we can express it as the condition that the first Fitting scheme is empty), as is the condition that it be semistable. If C is any fiber of the genus 1 fibration on X , then \mathcal{O}_C has a unique global section, and thus any invertible sheaf on C is simple. We thus obtain an algebraic space parametrizing semistable invertible sheaves on fibers of X . As before, any semistable invertible sheaf not supported on the special fiber rC_α is stable, and thus away from the special fiber, we recover the Néron model of the relative Pic^r . This fails on the special fiber for the simple reason that a given S -equivalence class can occur more than once. By the above classification of S -equivalence classes, we find that the S -equivalence class of M is determined by the orbit under twisting by ω_X of the invertible sheaf $M|_{C_\alpha}$; in particular, M is stable iff $M|_{C_\alpha} \cong \mathcal{O}_{C_\alpha}(p)$ for some point p of the smooth locus. Thus each S -equivalence class is represented by r distinct points of the algebraic space parameterizing semistable invertible sheaves.

The point is that we can compute tangent spaces in this algebraic space:

$$\begin{aligned} \dim \text{Ext}_X^1(M, M) &= \dim \text{Hom}_X(M, M) + \dim \text{Hom}_X(M, M \otimes \omega_X) \\ &= \dim \text{Hom}_{rC_\alpha}(M, M) + \dim \text{Hom}_{rC_\alpha}(M, M \otimes \omega_X) \\ &= \dim \Gamma(\mathcal{O}_{rC_\alpha}) + \dim \Gamma(\omega_X|_{rC_\alpha}) \\ &= 2, \end{aligned} \tag{7.8}$$

and thus the (2-dimensional) algebraic space is smooth at these points. Since twisting by ω_X acts without fixed points on the special fiber, it preserves tangent spaces, and thus the corresponding subset of the semistable moduli space is smooth. In particular, we conclude that the main component of the semistable moduli space is smooth on the locus represented by invertible sheaves, i.e., on the smooth locus of C'_α .

The minimal desingularization of the main component is thus a proper regular model of the relative Pic^r , so blows down to the minimal proper regular model. It follows that if we simply remove the singular points from the main component, the result maps to the minimal proper regular model. Now, the special fiber of the main component has the same number of components as the special fiber of the minimal proper regular model (which must have the same Kodaira type as C_α , [10, Thm. 5.3.1]). Thus the only way the surfaces can fail to be isomorphic is if the map from the minimal desingularization of the main component blows down a component of the original special fiber. (Indeed, we must blow down as many components as the minimal desingularization introduces.) Now, any section of $\text{Pic}^r(X \setminus C_\alpha)$ extends to a -1 -curve on the minimal proper regular model, which must in particular meet the special fiber in a point of the smooth locus. It follows that if the corresponding curve in the main component meets the special fiber in a point of the smooth locus, the corresponding component cannot be contracted. We will see that (under the additional saturation hypothesis) any component is met by some -1 -curve, giving the desired isomorphism.

Now, let e be any -1 -class on X which is transverse to C_α , and consider the corresponding τ -divisor $\tau(e)$. This certainly determines a well-defined curve in the complement of the special fiber (and any non-integral fibers), and we claim that its closure in the main component meets the special fiber in a single point, which lies in the smooth locus. Indeed, if M is supported on the special fiber and $\Gamma(M(-e)) \neq 0$, then we find that M is S -equivalent to $\mathcal{O}_{rC_\alpha}(e)$. Indeed, we may state the condition as $\text{Hom}(\mathcal{O}_{rC_\alpha}(e), M) = \text{Hom}(\mathcal{O}_X(e), M) \neq 0$. Since the image is both a quotient of the semistable sheaf $\mathcal{O}_{rC_\alpha}(e)$ and a subsheaf of the semistable sheaf M , the image is also semistable, and can be extended to Jordan-Hölder filtrations of both M and \mathcal{O}_{rC_α} . Since M

and $M \otimes \omega_X$ are S -equivalent, this is enough to completely determine the S -equivalence class of M as required.

In particular, we find that $\tau(e)$ meets the special fiber in the image of $e \cap C_\alpha$ in C'_α . (In particular, we may choose e so that this point lies in any desired component of C'_α .) It remains only to show that $\tau(e)$, or rather the corresponding invertible sheaf $\det R\Gamma(M(-e))$, is a -1 -class on the minimal proper regular model, and that this correspondence between -1 -classes extends to a homomorphism preserving the intersection pairing.

If $\mathcal{O}_C(e)$ is stable for every fiber C , then $\tau(e)$ consists precisely of sheaves of that form, and is thus a rational curve as required. Since it meets the generic fiber (and thus the anticanonical curve) in a single point, we conclude that it is a -1 -curve. More generally, adding a component of a nonspecial fiber to e does not change how $\tau(e)$ meets the special fiber or any integral fiber, and in this way we can arrange for $\mathcal{O}_C(e)$ to be stable for all C . In particular, we find that any class $\det R\Gamma(M(-e))$ obtained in this way is the sum of the class of a -1 -curve and a linear combination of components of nonspecial fibers.

It remains to see that this correspondence extends to a homomorphism and preserves the intersection pairing. Both of these are closed conditions on the (irreducible) moduli stack, so we may impose any dense conditions we desire. In particular, we may assume that C_α is smooth and every nonspecial fiber of X is integral, so that $\tau(e)$ is a -1 -curve for every -1 -class e . Let e' be another -1 -curve on X . Then $\tau(e') \cdot \tau(e)$ may be computed as the degree of $\det R\Gamma(M(-e'))|_{\tau(e)}$, or equivalently as the degree of $\det R\Gamma(\mathcal{O}_C(e - e'))$ as C varies over fibers of the genus 1 fibration on X . Now, consider the natural presentation

$$0 \rightarrow \mathcal{L}(e - e' + rK) \rightarrow \mathcal{L}(e - e') \rightarrow \mathcal{O}_C(e - e') \rightarrow 0 \quad (7.9)$$

Since $(e - e') \cdot K = 0$ and X has no -2 -curves, $e - e'$ is ineffective, and similarly for $e - e' + rK$, $e' - e + K$, $e' - e + (1 - r)K$. Thus $R\Gamma(\mathcal{O}_C(e - e'))$ is represented by the complex

$$H^1(\mathcal{L}(e - e' + rK)) \rightarrow H^1(\mathcal{L}(e - e')), \quad (7.10)$$

since the other cohomology groups vanish. This map depends linearly on the original map $\mathcal{L}(rK) \rightarrow \mathcal{O}_X$, and thus the desired degree may be computed as the common dimension of the two cohomology groups, which by Hirzebruch-Riemann-Roch is equal to $-1 - (e - e')^2/2 = e \cdot e'$ as required.

Finally, to see that this extends to a homomorphism, we note that the intersection form on the ten -1 -curves $s - e_1, f - e_1, e_1, \dots, e_8$ has determinant -1 , so the corresponding τ divisors span $\text{Pic}(X')$, just as the original divisors span $\text{Pic}(X)$. Since we may use the intersection form to expand any element of $\text{Pic}(X)$ in that basis, we conclude that the τ -divisor map is linear.

We excluded two cases above, in which C_α has Kodaira symbol I_8 or I_9 . In the latter case, we can easily see that any Jacobian fibration with an I_9 fiber has Weierstrass form $y^2 + txy + a_3y = x^3$ over the algebraic closure, with $a_3 \neq 0$. Any two such surfaces are geometrically isomorphic (in a nonunique way), and thus the claim follows immediately. Similarly, in the bad I_8 case, the corresponding Jacobian fibration must be the desingularization of the blow-up in the identity of a surface

$$y^2 + txy = x^3 + a_2x^2 + a_4x \quad (7.11)$$

with $a_4 \neq 0$; two such surfaces are geometrically isomorphic iff they have the same value of a_2^2/a_4 . There are two -1 -curves on this surface that do not meet the singular point, which meet the corresponding G_m in the points $\lambda, 1/\lambda$ where

$$\frac{(\lambda + 1)^2}{\lambda} = \frac{a_2^2}{a_4}. \quad (7.12)$$

In particular, the surface is determined up to (again nonunique) isomorphism by the two points of intersection, and again the claim follows. \square

Remark. Note that in the good cases, we prove a slightly stronger fact: not only is the special fiber of $\text{Pic}^r(X)$ isomorphic to C'_α , but the isomorphism we construct is compatible with the isomorphism $X' \cong \text{Pic}^r(X)$. This presumably still holds in the I_8 and I_9 cases, but the above calculation does not suffice.

Remark. We should note that by the remark following Proposition 6.3, there is no universal family over the stable locus of this moduli space.

The case that C_α is nonreduced appears to be more subtle. Since $\text{Pic}^0(C_\alpha) \cong G_a$, this case (for $r > 1$) only arises in characteristic p , with $r = p$. We still have an action of $\text{Pic}^0(C_\alpha)$ on C_α , which restricts to an action of the étale subscheme generated by $K|_{C_\alpha}$. Since this action fixes the singular points of C_α , it must act trivially on any component appearing with multiplicity. In the simplest case, I_0^* , this would indicate that $C'_\alpha \cong C_\alpha$. Indeed, a curve of type I_0^* is determined by the cross-ratio λ of the four points where the double component meets the reduced components, or, more precisely, by the invariant function

$$\frac{j}{256} = \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}. \quad (7.13)$$

Since the group acts trivially on the special fiber, the quotient should preserve the cross-ratio, but experiments (for $p = 2$, $p = 3$) instead suggest that the true special fiber of the moduli space satisfies $\lambda(C'_\alpha) = \lambda(C_\alpha)^p$.

The one disadvantage of considering Pic^r is that the corresponding morphisms B are never maps of trivial bundles; as a result, we cannot directly apply statements about Pic^r in the non-commutative setting. With this in mind, we consider the corresponding moduli space of semistable sheaves of Euler characteristic 0. The individual fibers are not too difficult to understand, but to understand the full moduli space, we will need to work étale locally, and thus must to some extent consider more general genus 1 fibrations (e.g., with total space which is not rational). For this, we note first that the case $r = 1$ of the above result is easy to extend to arbitrary elliptic surfaces.

Theorem 7.3. *Let $\psi : X \rightarrow C$ be a smooth, relatively minimal, genus 1 fibration with no multiple fibers, and let $\mathcal{M}_X(1)$ be the moduli space classifying stable sheaves M of Euler characteristic 1 and with $c_1(M)$ a fiber of ψ . Then there is a natural isomorphism $\mathcal{M}_X(1) \cong X$.*

Proof. We first note that Lemma 7.2, though stated for anticanonical curves on rational surfaces, applies equally well (with essentially the same argument) to curves of canonical type (i.e., curves with the same intersection matrix as a fiber of a minimal proper regular model of a genus 1 fibration) on surfaces, so in particular to fibers of ψ . In particular, the construction there of sheaves $\mathcal{O}_C(p)$, and the proof that those sheaves are stable, carries over directly. In particular, any stable sheaf in $\mathcal{M}_X(1)$ has this form, and we can easily construct a corresponding universal sheaf on $X \times X$. \square

For Jacobian fibrations, the Euler characteristic 0 case is also relatively straightforward to deal with.

Corollary 7.4. *Let $\psi : X \rightarrow C$ be a smooth, relatively minimal, genus 1 fibration with a section $s : C \rightarrow X$, and let $\mathcal{M}_X(0)$ be the moduli space classifying stable sheaves M of Euler characteristic 0 and with $c_1(M)$ a fiber of ψ . Then there is a natural birational morphism $X \rightarrow \mathcal{M}_X(0)$ for which the exceptional locus is the union of all vertical curves not meeting the section.*

Proof. To construct a morphism from $X \cong \mathcal{M}_X(1)$ to $\mathcal{M}_X(0)$, it will suffice to give a construction taking stable sheaves of Euler characteristic 1 to semistable sheaves of Euler characteristic 0. In particular, let M be the sheaf corresponding to a point of $\mathcal{M}_X(1)$, and consider the twist $M \otimes \mathcal{L}(-s)$ of Euler characteristic 0. Any subsheaf of this twist has the form $M' \otimes \mathcal{L}(-s)$ for some $M' \subset M$; since s meets fibers transversely, we have

$$\chi(M' \otimes \mathcal{L}(-s)) \leq \chi(M') \leq 0, \quad (7.14)$$

where the second inequality follows from stability of M . This, of course, is precisely the inequality we needed to show to demonstrate semistability of M .

Now, if $M \otimes \mathcal{L}(-s)$ is stable, then it is determined by its corresponding point in $\mathcal{M}_X(0)$, so that we can recover M from its image; thus if $M \otimes \mathcal{L}(-s)$ is stable, then it falls outside the exceptional locus.

Write $M = \mathcal{O}_f(p)$ with f a fiber of ψ and $p \in F$ a closed point. If f is integral, then $M \otimes \mathcal{L}(-s)$ is automatically stable (as a torsion-free sheaf with integral support). In addition, if p is a smooth point of the identity component (the component meeting s) of f , then $M \otimes \mathcal{L}(-s)$ is an invertible sheaf of degree 0 on every component. It follows that twisting by $M \otimes \mathcal{L}(-s)$ preserves stability; since \mathcal{O}_f is stable, so is $M \otimes \mathcal{L}(-s)$ in this case.

If p is not a point of the identity component, then $M \otimes \mathcal{L}(-s)$ is invertible near the identity component, and has degree -1 on that component. We thus have a morphism

$$M \otimes \mathcal{L}(-s) \rightarrow \mathcal{O}_{f_0}(-1), \quad (7.15)$$

where f_0 is the identity component. This makes M strictly semistable. By semicontinuity, such a morphism continues to exist on the closure of the complement of the identity component.

In other words, $M \otimes \mathcal{L}(-s)$ is strictly semistable iff p lies on some nonidentity component of f . It remains to show that these components are contracted, and that the morphism is dominant. By dimensionality, it will be enough to show that any invertible sheaf in $\mathcal{M}_X(0)$ has degree 0 on every component (so remains stable upon twisting by the stable sheaf $\mathcal{O}_f \otimes \mathcal{L}(s)$), and that each fiber of $\mathcal{M}_X(0)$ has at most one strictly semistable sheaf. This follows from the next Lemma. \square

Remark. In particular, $\mathcal{M}_X(0)$ is the Weierstrass model of X .

Lemma 7.5. *Let C be a curve of canonical type on a surface X , and let M be a stable sheaf of Euler characteristic 0 set-theoretically supported on C . Then either M is an invertible sheaf on C , of degree 0 on every component of C , or M is the direct image of $\mathcal{O}_{\mathbb{P}^1}(-1)$ under some morphism $\mathbb{P}^1 \rightarrow C$.*

Proof. Note that since C is orthogonal to every component of C , $M \otimes \mathcal{L}(-C)$ is also stable, and thus the natural map $M \otimes \mathcal{L}(-C) \rightarrow M$ must be 0; it follows that M is scheme-theoretically supported on C .

If $c_1(M)$ is not a multiple of C , then signature considerations show $c_1(M)^2 < 0$ and thus, since $c_1(M)$ is a sum of components of C , that there exists a component C_1 of C such that $C_1 \cdot c_1(M) < 0$. In that case, we find

$$\dim \operatorname{Hom}(\mathcal{O}_{C_1}(-1), M) - \dim \operatorname{Ext}^1(\mathcal{O}_{C_1}(-1), M) + \dim \operatorname{Ext}^2(\mathcal{O}_{C_1}(-1), M) = -C_1 \cdot c_1(M) > 0 \quad (7.16)$$

Since M has canonical type, C_1 is a -2 -curve, so orthogonal to the canonical class of X , and thus $\dim \operatorname{Ext}^2(\mathcal{O}_{C_1}(-1), M) = \dim \operatorname{Hom}(M, \mathcal{O}_{C_1}(-1))$. We thus conclude that there is a morphism between M and $\mathcal{O}_{C_1}(-1)$; since M is stable, this can only occur if $M \cong \mathcal{O}_{C_1}(-1)$.

Thus, suppose $c_1(M) = rC$, and consider the sheaf $M \otimes \mathcal{O}_C(p)$ for some smooth point $p \in C$, of Euler characteristic r . If this has a nonzero quotient of nonpositive Euler characteristic, then twisting by the ideal sheaf of p gives a nonzero proper quotient of M of nonpositive Euler characteristic, contradicting stability. But then by the proof of Lemma 7.2, there is a surjective morphism

$$M \otimes \mathcal{O}_C(p) \rightarrow \mathcal{O}_C(p') \quad (7.17)$$

for some point $p' \in C$, giving a surjection

$$M \rightarrow \mathcal{O}_C(p') \otimes \mathcal{I}_p. \quad (7.18)$$

Since M is stable and both sheaves have Euler characteristic 0, this must in fact be an isomorphism, and we have already seen that such sheaves are strictly semistable unless p' does not lie on any nonidentity component of C . Thus either M is invertible on C , of degree 0 on every component, or C is integral and p' is singular on C ; in the latter case, the normalization of C is \mathbb{P}^1 , and the sheaf must be the direct image of $\mathcal{O}_{\mathbb{P}^1}(-1)$ on the normalization. \square

Corollary 7.6. *Let $\psi : X \rightarrow C$ be a smooth, relatively minimal, genus 1 fibration with no multiple fibers, and let $\mathcal{M}_X(0)$ be the moduli space classifying stable sheaves M of Euler characteristic 0 and with $c_1(M)$ a fiber of ψ . Then $\mathcal{M}_X(0)$ is naturally isomorphic to the Weierstrass model of the relative Jacobian of X .*

Proof. If X has a section, we are done; in general, such a section exists étale locally, allowing us to identify $\mathcal{M}_X(0)$ with the Weierstrass model étale locally on C . Since these local identifications are natural, they descend to give a natural isomorphism as required. \square

Theorem 7.7. *Under the hypotheses of Theorem 7.1, let $\mathcal{M}_X(0)$ denote the moduli space of semistable 1-dimensional sheaves with $c_1(M) = -rK_X$, $\chi(M) = 0$. Then $\mathcal{M}_X(0)$ is isomorphic to the Weierstrass model of the relative Pic^r as computed above.*

Proof. Choose a -1 -curve e on X , and consider the map $M \rightarrow M \otimes \mathcal{L}(-e)$ from the moduli space of semistable sheaves with $\chi(M) = r$ to $\mathcal{M}_X(0)$. Note that this is in general only a rational map, since for sheaves on reducible fibers, the image could easily be unstable. However, this map is certainly well-defined for sheaves on smooth fibers, and more importantly, is well-defined on the special fiber. Since extensions of semistable sheaves of Euler characteristic 0 are semistable, it is sufficient to prove that $M' \otimes \mathcal{L}(-e)$ is semistable for M' stable of the same slope as M . But this implies $\chi(M') = 1$, which we have already seen suffices for stability.

We thus see that $\mathcal{M}_X(0)$ is naturally isomorphic to the Weierstrass model in a neighborhood of the special fiber. Since we also have such an identification on the complement of the special fiber (as the complement of the unique multiple fiber has no multiple fibers), the claim follows. \square

Remark. In particular, given a choice of -1 -curve e on X such that the corresponding τ divisor is irreducible, $\mathcal{M}_X(0)$ is obtained from X' by contracting all -2 -curves disjoint from e .

8 Elliptic difference equations

We now wish to translate the above theory back to the realm of difference equations. From a geometric perspective, the simplest case is that of symmetric elliptic difference equations, since then not only is the surface smooth, but so is the anticanonical curve. If C is a smooth genus 1 curve, then the above considerations tell us that symmetric difference equations on C twisted by a

line bundle are in natural correspondence with triples (X_0, ϕ, M_0) where X_0 is a Hirzebruch surface (in particular with specified map to \mathbb{P}^1), M_0 is a sheaf on X_0 with $H^0(M_0) = H^1(M_0) = 0$, and $\phi : C \rightarrow X_0$ embeds C as an anticanonical curve. (As we mentioned above, this is not quite correct, as these sheaves also include degenerate cases where some of the singularities cancel each other.) Moreover, the pairs (X_0, ϕ) are classified by elements of $\text{Pic}^2(C) \times \text{Pic}^2(C)$ or $\text{Pic}^1(C) \times \text{Pic}^2(C)$, depending on the parity of the Hirzebruch surface, or equivalently depending on the parity of the degree of the twisting line bundle.

Since $C \cong C_\alpha$ is smooth, $M_0|_{C_\alpha}$ is a direct sum of structure sheaves of jets. If X is the minimal desingularization of the blowup of X_0 in those jets, then there is a natural way to lift M_0 to a sheaf M on X which is disjoint from the anticanonical curve. (This is the minimal lift of [42], see in particular Proposition 6.12 there.) In this way, we encode the singularities of the equation in the surface X and the Chern class of M .

As above, the surface is determined by the classes $\phi^*(s)$, $\phi^*(f)$ and $\phi^*(e_i)$ for $1 \leq i \leq m$; since C is smooth, the bundles $\phi^*(e_i)$ can be identified with line bundles $\mathcal{L}(p_i)$ for points p_i . Now, consider the extension

$$B : \rho^*V \rightarrow \rho^*W \otimes \mathcal{L}(s) \quad (8.1)$$

of our original B to X_0 . The Chern class of M_0 is given by that of $\det(B)$, so has the form $ns + df$ where $n = \text{rk}(W)$ is the order of the difference equation. At any point $p \in C_\alpha$, we can view B as a matrix over the local ring $\mathcal{O}_{C_\alpha, p}$. Up to left- and right-multiplication by invertible matrices, we can diagonalize B , and then define a partition $\lambda(B; p)$ by letting $\lambda_j(B; p)$ for $j \geq 1$ be the number of diagonal elements contained in \mathfrak{m}^j . Local computations then tell us that $\lambda_1(B; p)$ is the rank of M_0 at p , and $\lambda_j(B; p)$ is the rank after blowing up p $j - 1$ times. If $e_{p,j}$ denotes the j -th class in the sequence e_1, \dots, e_m such that $\phi^*(e_i) \cong \mathcal{L}(p)$, then we find

$$c_1(M) = c_1(M_0) - \sum_{p,i} \lambda_i(B; p) e_{p,i}. \quad (8.2)$$

(Note that we must blow up p at least as many times as there are parts of λ in order to make the resulting sheaf disjoint from the anticanonical curve.)

We can also describe the invariants $\lambda(B; p)$ in terms of the original shift matrix A . If p is not fixed by η , then we can again diagonalize A over the local ring at p (by left- and right- multiplication). The resulting equivalence classes are given by weights of GL_n , i.e., nonincreasing sequences of integers. We then find that $\lambda(B; p)$ is determined by the positive coefficients of this weight; the negative coefficients of the weight appear in $\lambda(B; \eta(p))$. When p is fixed by η , the situation is more complicated; up to the relevant equivalence relation (left multiplication by invertible matrices over the local ring, right multiplication by symmetric invertible matrices over the local field), B is a direct sum of matrices

$$(1), \quad (u), \quad \begin{pmatrix} 1 & u \\ 0 & u^e \end{pmatrix}, e > 1. \quad (8.3)$$

(This needs to be adjusted slightly when the equation is twisted by a line bundle.) The second case is a singularity of order 1, but corresponds to an eigenvalue -1 of A (assuming the characteristic is not 2); the third cases have order e , but appear to have order $e - 1$. (In characteristic 2, this phenomenon is worse: singularities of order e appear to have order $\max(e - 2, 0)$ or $\max(e - 4, 0)$ depending on whether C is ordinary or supersingular.)

Of course, a sheaf M on X disjoint from C_α need not come from a maximal morphism B . The condition $\text{Hom}(M, \mathcal{O}_g(-1)) = \text{Hom}(\mathcal{O}_g(-1), M) = 0$ for every component g of a fiber is simplified by disjointness, since we need only consider those g which are disjoint from C_α . In particular, g must be a -2 -curve, and since it is contained in a fiber, must be a root of the D_m subsystem.

The subquotient corresponding to the standard representation of a difference equation will differ from M by a number of copies of sheaves $\mathcal{O}_g(-1)$, and thus in particular correspond to strictly semistable points of the moduli space. As a result, the specific extension classes will be irrelevant, and we thus obtain precisely one point of the moduli space for each difference equation that arises.

The constraint on the difference equations underlying such semistable points is that their Chern class must differ from the specified Chern class by a nonnegative sum of -2 -curves disjoint from C_α . This can be translated in terms of the local data $\lambda(B; p)$ as follows. Subtracting roots of the form $e_i - e_j$ simply replaces the partitions $\lambda(B; p)$ by partitions of the same size which cover it in the dominance ordering. Roots of the form $f - e_i - e_j$ either subtract 1 from the first parts of both $\lambda(B; p)$ and $\lambda(B; \eta(p))$ or (when $p = \eta(p)$) subtract 1 from the first two parts of $\lambda(B; p)$. For $p \neq \eta(p)$, the conditions combine to say that the relevant weight of GL_n (related to the conjugate partitions) becomes smaller in dominance order, with something similar in the case of ramification points.

One special case we should note is that when $D = s - f$ (assuming this is effective), then as usual, the moduli space is a point (the sheaf $\mathcal{O}_{s-f}(-1)$), and the corresponding difference equation is just the trivial equation $v(z + q) = v(z)$. More generally, if $s - f$ is effective and $D \cdot (s - f) < 0$, then the corresponding difference equation will have a block-triangular structure such that the first or last block is trivial.

Of course, a sheaf on an anticanonical surface X with $C_\alpha \cong C$ does not determine a difference equation unless we also choose a blowdown structure (more precisely, a choice of blowdown structure modulo the action of ineffective roots of the A_m subsystem). In other words, a given moduli space of sheaves corresponds to many different moduli spaces of difference equations, one for each blowdown structure on the surface. In addition, we can also twist by line bundles and apply the duality $\mathcal{E}xt^1(-, \omega_X)$. The latter is canonical, but to make sense of the former requires a choice of blowdown structure. Thus in the generic situation, we have an action of $\mathrm{Aut}(C) \times (W(E_{m+1}) \times Z_2) \ltimes \mathbb{Z}^{m+2}$, where the cyclic group Z_2 acts by duality; in the nongeneric situation, there is a partial action taking into account the usual issues with effective reflections. Since the group $W(E_{m+1})$ simply acts on the set of ways of interpreting sheaves, it certainly respects the Poisson structure on the moduli space, and the construction of the Poisson structure implies that it is preserved by twisting. Duality is anti-Poisson ([42, Prop. 7.12]), and $\mathrm{Aut}(C)$ acts on the Poisson structure in the same way it acts on holomorphic differentials. In particular, $\mathrm{Pic}^0(C) \subset \mathrm{Aut}(C)$ preserves the Poisson structure, and hyperelliptic involutions are anti-Poisson; in the $j = 0$ and $j = 1728$ cases, we also have automorphisms multiplying the Poisson structure by other roots of unity. In terms of moduli spaces of difference equations, each of these operations will change the parameters (the twisting line bundle, the points with allowed singularities, q), but should give birational maps between the corresponding moduli spaces. The action on q is essentially forced: Poisson maps should preserve q , while anti-Poisson maps should negate q (and for $j = 0$, $j = 1728$, $\mathrm{Aut}(C)$ acts as one would expect on $q \in \mathrm{Pic}^0(C)$).

Note that the subgroup $D^\perp \subset \mathbb{Z}^{m+2}$ acts as a (large) abelian group of rational Poisson automorphisms of the moduli spaces $\mathcal{Irr}(D, x)$. (The element K_X acts trivially, of course, as does any effective class in D^\perp .) In particular, they give rise to a discrete integrable system acting on a rational variety, which relative to the fibration by $\mathrm{supp}(M)$ acts by translation within each fiber (a torsor over the Jacobian of the support). Similarly, we will describe a q -twisted version of this action, which appears to be an analogue of (higher-order) discrete Painlevé equations, a non-autonomous version of translation on an abelian variety.

Of course, in our setting, we only have a relaxation of the true moduli spaces of difference equations, but will still gain insight by looking at how simple reflections, twists by basis elements,

and duality act; in each case, there will be an obvious way to take into account the shifting by q . Since the simplest operations do not preserve the triviality condition $H^0(M) = H^1(M) = 0$ even generically (since they do not preserve the condition $\chi(M) = 0$), we need to allow W to be nontrivial. We thus note (per [2] and [44]) that in the (analytic) difference equation case, B corresponds to a morphism

$$B : \pi_{\eta'}^* V \rightarrow \pi_{\eta}^* W \otimes \mathcal{L} \quad (8.4)$$

of bundles on \mathbb{C}/Λ , where η is the involution $z \mapsto -q - z$, η' is the involution $z \mapsto -z$, and \mathcal{L} is the twisting line bundle. We do, however, assume V maximal where convenient, since this is in any event the main case of interest (and it is easy enough to figure out what goes wrong when maximality fails).

The simplest operation is twisting by $\mathcal{L}(f)$, which simply twists the bundles V and W by $\mathcal{O}_{\mathbb{P}^1}(1)$. This is also easy to extend to an action on difference equations: the only change is that since V and W are pulled back through different degree 2 maps, we must absorb the difference into \mathcal{L} , thus changing the twisting bundle by the element of $\text{Pic}^0(C)$ corresponding to q . (In other words, twisting by f changes $\phi^*(s)$ by q .)

Although the operation $M \mapsto M \otimes \mathcal{L}(s)$, or equivalently $M_0 \mapsto M_0 \otimes \mathcal{L}(s)$, is just as natural in terms of sheaves, the translation to morphisms of vector bundles on C is quite a bit more subtle, for the simple reason that twisting does not respect the resolution we are using. Now, we can write the original morphism $B : \pi_{\eta}^* \rightarrow \pi_{\eta}^* W \otimes \phi^* \mathcal{L}(s)$ in the form

$$B = B_0(x, w)y_0 + B_1(x, w)y_1 \quad (8.5)$$

where x, w are homogeneous coordinates on $\mathbb{P}^1 \cong \pi_{\eta}(C) \cong \rho(X)$, and y_0, y_1 generate $\pi_{\eta*} \phi^* \mathcal{L}(s) \cong \rho_* \mathcal{L}(s)$, viewed as a graded module over $k[x, w]$. (In the untwisted case, we used $\phi^* \mathcal{L}(s_{\min})$, but this just differs by twisting by f ; i.e., we assume that the twisting bundle has degree 2 when its degree is even.) Now, since B comes from the standard resolution of M_0 , we can twist by $\mathcal{L}(s)$ and take direct images to obtain a short exact sequence

$$0 \rightarrow V \xrightarrow{(B_0, B_1)} W \otimes \rho_* \mathcal{L}(s) \xrightarrow{(B'_1, -B'_0)} \rho_*(M_0 \otimes \mathcal{L}(s)) \rightarrow 0 \quad (8.6)$$

where B'_1, B'_0 are suitable morphisms of sheaves on \mathbb{P}^1 . The sheaf $\rho_*(M_0 \otimes \mathcal{L}(s))$ is torsion-free, since otherwise $\text{Hom}(\mathcal{O}_f(-1), M_0)$ would be nonzero for some fiber f , contradicting maximality of V . In particular, we have

$$W' \cong \rho_*(M_0 \otimes \mathcal{L}(s)). \quad (8.7)$$

We also have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \rho^* V & \longrightarrow & \rho^* W \otimes \rho^* \rho_* \mathcal{L}(s) & \longrightarrow & \rho^* \rho_*(M_0 \otimes \mathcal{L}(s)) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \rho^* V & \longrightarrow & \rho^* W \otimes \mathcal{L}(s) & \longrightarrow & M_0 \otimes \mathcal{L}(s) \longrightarrow 0 \end{array} \quad (8.8)$$

with exact rows; since each sheaf in the bottom sequence is ρ -globally generated, the vertical morphisms are surjective, and have isomorphic kernels. We thus find that V' fits into an exact sequence

$$0 \rightarrow \rho^*(V') \otimes \mathcal{L}(-s) \rightarrow \rho^* W \otimes \rho^* \rho_* \mathcal{L}(s) \rightarrow \rho^* W \otimes \mathcal{L}(s) \rightarrow 0. \quad (8.9)$$

It follows that

$$V' \cong \begin{cases} W & s^2 = 0 \\ W \otimes \mathcal{L}(-f) & s^2 = -1 \end{cases} \quad (8.10)$$

We moreover find that (apart from this twisting), the map from V' to W' is simply given by $B'_1 y_1 + B'_0 y_0$. To avoid the issue with twisting, we will compute $M \otimes \mathcal{L}(s + f)$ in the odd case.

We now observe that, viewing this as a morphism B' on C , we have

$$B' \eta^* B - \eta^* (B' \eta^* B) = (B'_0 B_1 - B'_1 B_0)(y_0 \eta^* y_1 - y_1 \eta^* y_0) = 0, \quad (8.11)$$

since $B'_0 B_1 = B'_1 B_0$ by construction. Since $B' \eta^* B$ is η^* -invariant, we can use it to modify the factorization of A to obtain

$$A = (\eta^* B)^{-t} B^t = (B')^t (\eta^* B')^{-t}, \quad (8.12)$$

and find that the new A has the form

$$A' = (\eta^* B')^{-t} (B')^t = (B')^{-t} A (B')^t. \quad (8.13)$$

It is somewhat more natural to express the inverse of this operation.

Proposition 8.1. *Let $A(z)$ be a twisted elliptic matrix with $\eta^* A = A^{-1}$, twisted by a line bundle of degree δ , and let M be the corresponding sheaf. Then the twisted sheaf $M \otimes \mathcal{L}(-s - (2 - \delta)f)$ corresponds to the matrix $B^t A B^{-t}$, where B comes from the minimal factorization of A .*

This operation need only be modified very slightly (since conjugation should become a gauge transformation) to make sense for difference equations: if we start with the system

$$v(z + q) = B(-q - z)^{-t} B(z)^t v(z) \quad v(-z) = v(z), \quad (8.14)$$

we simply want the equations satisfied by $w(z) = B(z)^t v(z)$, namely

$$w(-q - z) = w(z) \quad B(-z)^{-t} w(-z) = B(z)^{-t} w(z). \quad (8.15)$$

The only nonobvious point is that this new equation is symmetric with respect to a slightly different involution; we will see this phenomenon naturally arising in the noncommutative setting. (Here twisting by s changes $\phi^*(f)$ by q ; it also changes $\phi^*(s)$ when $s^2 = -1$. In general, the rule is that twisting by D changes ϕ^* by $(D \cdot -)q$; this is the only $W(E_{m+1})$ -invariant rule compatible with what we have so far seen.) This can also be sidestepped by choosing a point “ $q/2$ ” such that $2(q/2) = q$, and replacing the above w by $w(z) = B(z - q/2)^t v(z - q/2)$.

The next operation we consider is duality, as this can also be computed on the Hirzebruch surface. Indeed, by [42, Prop. 7.11], the minimal lift operation commutes with the canonical duality, so we just need to understand the dual of M_0 . Applying $\mathbf{R}\mathcal{H}om(-, \omega_X)$ to the standard presentation

$$0 \rightarrow \rho^* V \otimes \mathcal{L}(-s) \xrightarrow{B} \rho^* W \rightarrow M_0 \rightarrow 0 \quad (8.16)$$

gives

$$0 \rightarrow \mathcal{H}om(\rho^* W, \omega_X) \xrightarrow{B^t} \mathcal{H}om(\rho^* V, \omega_X) \otimes \mathcal{L}(s) \rightarrow \mathcal{E}xt^1(M_0, \omega_X) \rightarrow 0. \quad (8.17)$$

Now, we have

$$\mathcal{H}om(\rho^* W, \omega_X) \cong \rho^* \mathcal{H}om(W, \mathcal{O}_{\mathbb{P}^1}) \otimes \omega_X \quad (8.18)$$

and $\omega_X \cong \mathcal{L}(-2s - (4 - \delta)f)$, where $\delta \in \{1, 2\}$ is the degree of the twisting bundle. We thus see that this is a presentation of the alternate kind we just considered, and can thus determine the corresponding relaxed difference equation. We thus find that $\mathcal{E}xt^1(M, \omega_X) \otimes \mathcal{L}(2f)$ corresponds to the matrix

$$A' = B(\eta^* B)^{-1} = A^t = \eta^* A^{-t}. \quad (8.19)$$

In other words, dual sheaves correspond (up to the involution) to adjoint difference equations. More precisely, if we both dualize and act by η , we obtain the adjoint equation

$$w(z + q) = A(z)^{-t} w(z); \quad (8.20)$$

the dual sheaf itself corresponds to

$$w(z - q) = A(z - q)^t w(z), \quad (8.21)$$

which is of course precisely the same equation viewed as a $-q$ -difference equation. Note that the additional $2f$ twist is precisely what we need in order for the $\chi(M \otimes \mathcal{L}(-f)) = 0$ condition to be preserved by duality.

We next turn to twists by e_i . From [42, Cor. 6.7], we find that the action of such twists on M_0 has the following form. If $M' \cong M \otimes \mathcal{L}(-e_i)$, then M' is acyclic for $\pi : X \rightarrow X_0$, and its direct image M'_0 fits into a short exact sequence

$$0 \rightarrow M'_0 \rightarrow M_0 \rightarrow \mathcal{O}_p^r \rightarrow 0 \quad (8.22)$$

where p is the point of X_0 lying under e_i , and the morphism $M_0 \rightarrow \mathcal{O}_p^r$ is suitably canonical (with $r = c_1(M) \cdot e_i$). Local computations let us describe this morphism in the elliptic case. First, if e_i arises from the first time we blow up p , then it is just the canonical morphism

$$M_0 \rightarrow \text{Hom}_k(\text{Hom}(M_0, \mathcal{O}_p), \mathcal{O}_p). \quad (8.23)$$

More generally, the structure of B over the local ring induces a natural increasing filtration F_l of $\rho^*W \otimes \mathcal{O}_p^n$, induced by tensor product from the filtration

$$F_l^+ = \text{im}(u^{1-l}B) \cap \rho^*W \quad (8.24)$$

where u is a uniformizer. If e_i arises from the l -th time we blow up p , then the corresponding morphism $M_0 \rightarrow \mathcal{O}_p^r$ is induced by the morphism

$$\rho^*W \rightarrow \rho^*W \otimes \mathcal{O}_p \cong F_\infty \rightarrow F_\infty/F_l. \quad (8.25)$$

Once we have identified the map $M_0 \rightarrow \mathcal{O}_p^r$, we then need to understand how the new A' is related to the original A . We first note that M'_0 is indeed ρ_* -acyclic. Otherwise, there is a nonzero morphism from M'_0 to $\mathcal{O}_f(-2)$ for some fiber f , so that the short exact sequence defining M'_0 pushes forward to an extension

$$0 \rightarrow \mathcal{O}_f(-2) \rightarrow F \rightarrow \mathcal{O}_p^r \rightarrow 0. \quad (8.26)$$

Since $\text{Hom}(M_0, \mathcal{O}_f(-2)) = 0$, this is a non-split extension, so $p \in f$; but then $F \cong \mathcal{O}_f(-1) \oplus \mathcal{O}_p^{r-1}$, contradicting the fact that $\text{Hom}(M_0, \mathcal{O}_f(-1)) = 0$ (since $M_0 \otimes \mathcal{L}(-s)$ is ρ_* -acyclic). We also find that $W' = \rho_*M'_0$ is torsion-free, since $\text{Hom}(\mathcal{O}_f, M'_0) \subset \text{Hom}(\mathcal{O}_f, M_0) = 0$ for any fiber f .

Since M'_0 is ρ_* -acyclic, we can take the direct image of the defining extension to obtain a short exact sequence

$$0 \rightarrow W' \xrightarrow{D_1} W \rightarrow \mathcal{O}_{\rho(p)}^r \rightarrow 0, \quad (8.27)$$

for a suitable morphism D_1 , and since ρ is flat the inverse image is also exact:

$$0 \rightarrow \rho^*W' \rightarrow \rho^*W \rightarrow \mathcal{O}_f^r \rightarrow 0, \quad (8.28)$$

where f is the fiber containing p . We thus obtain a map of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \rho^*V(-s) & \xrightarrow{B} & \rho^*W & \longrightarrow & M_0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_f(-1)^r & \longrightarrow & \mathcal{O}_f^r & \longrightarrow & \mathcal{O}_p^r \longrightarrow 0, \end{array} \quad (8.29)$$

in which the second and third vertical maps are surjective. The snake lemma tells us that the natural four-term exact sequence

$$0 \rightarrow \rho^*V'(-s) \rightarrow \rho^*W' \rightarrow M'_0 \rightarrow \mathcal{O}_f(-1)^{r'} \rightarrow 0 \quad (8.30)$$

(the cokernel has the form $\mathcal{O}_f(-1)^{r'}$ since this is the only extent to which M'_0 can fail to be ρ -globally generated) is related to a four-term exact sequence

$$0 \rightarrow \rho^*V'(-s) \rightarrow \rho^*V(-s) \rightarrow \mathcal{O}_f(-1)^r \rightarrow \mathcal{O}_f(-1)^{r'} \rightarrow 0. \quad (8.31)$$

This is the twist of the inverse image of an exact sequence

$$0 \rightarrow V' \xrightarrow{D_2} V \rightarrow \mathcal{O}_p^r \rightarrow \mathcal{O}_p^{r'} \rightarrow 0. \quad (8.32)$$

Of course, if $r' \neq 0$, then M'_0 is not globally generated, so we should really replace M'_0 by the generated subsheaf; in this case, the twisting operation will not be invertible, but a finite amount of such twisting will suffice to remove any components of M'_0 supported on fibers. In any event, the new B can be written as

$$B' = D_1^{-1}BD_2, \quad (8.33)$$

and thus, since D_1 and D_2 are η -invariant,

$$A' = D_1^t A D_1^{-t}. \quad (8.34)$$

Again, this conjugation should become a gauge transformation: if $w(z) = D_1(z)^t v(z)$, then w satisfies the equation

$$w(z+q) = D_1(z+q)^t A(z) D_1(z)^{-t} w(z). \quad (8.35)$$

This gauge transformation has the effect of shifting the singularity at p by q (as we expect from how twisting should affect ϕ^*); in terms of the invariants $\lambda(B;p)$, it moves the appropriate part to the partition corresponding to $p-q$. The case $r' \neq 0$ corresponds to a situation in which the shifted singularity ends up cancelling an existing singularity at $p-q$. (In particular, we see that the sheaves with components supported on fibers correspond to equations with apparent singularities, that is to say singularities which can be removed by a suitable gauge transformation.)

Remark. A similar construction (unfortunately called “elementary transformations”) for sheaves on \mathbb{P}^2 was given in [56].

We note that since $M \otimes \mathcal{O}_{C_\alpha} = 0$, twisting by

$$\mathcal{L}(C_\alpha) \cong \mathcal{L}(2s + 2f - \sum_i e_i) \quad (8.36)$$

has no effect on the sheaf. Since for difference equations, the various twisting operations all change various parameters by multiples of q , this cannot quite be true for difference equations; instead, twisting by the canonical class simply shifts z by q .

Remark. With the above constructions in mind, we can also easily identify the various gauge transformations of [46] (called isomonodromy transformations there) with twists; in particular, the gauge transformations with matrices described by [46, Thm. 4.6] are twists by $s + f - \sum_{1 \leq i \leq m+3} e_i$.

It remains to understand how the group $W(E_{m+1})$ acts. The S_m subgroup is of course easy to understand, as it simply changes the order in which we blow up the distinct singular points. Indeed, since it does not change the final Hirzebruch surface, we should not expect it to have any effect on the interpretation of the sheaf.

To understand the subgroup $W(D_m)$, it will be enough to understand how the elementary transformation acts, as it conjugates the two S_m subgroups in $W(D_m)$. We suppose now that X_0 is even, since of course the odd to even elementary transformation is just the inverse. And of course, since the elementary transformation does not change the blowdown structure past X_1 , it suffices to consider the direct image M_1 of M on X_1 . Let e be the exceptional curve on X_1 over X_0 , and let X'_0 be the transformed Hirzebruch surface. The nature of the minimal lift operation implies that we have a short exact sequence of the form

$$0 \rightarrow \mathcal{O}_e(-1)^r \rightarrow \pi^* M_0 \rightarrow M_1 \rightarrow 0. \quad (8.37)$$

If we take the direct image under π' (the map that blows down the complement e' of e in its fiber), then we see that $\mathcal{O}_e(-1)$ is acyclic with direct image of the form $\mathcal{O}_f(-1)$, and thus

$$0 \rightarrow \mathcal{O}_f(-1)^r \rightarrow \pi'_* \pi^* M_0 \rightarrow \pi'_* M_1 \rightarrow 0 \quad (8.38)$$

is exact. This, of course, is precisely the situation we encounter with non-maximal splittings; in particular, we can compute the new matrix A' equally well from either $\pi'_* M_1$ or $\pi'_* \pi^* M_0$. Now,

$$\rho'_* \pi'_* \pi^* M_0 \cong \rho_* \pi_* \pi^* M_0 \cong \rho_* M_0, \quad (8.39)$$

and thus $W' \cong W$. Similarly, the nonmaximal bundle V'' can be computed (up to a scalar) by

$$\begin{aligned} \rho'_* (\pi'_* \pi^* M_0 \otimes \mathcal{L}(-s' - f)) &= \rho'_* \pi'_* (\pi^* M_0 \otimes \mathcal{L}(e_1 - s - f)) \\ &= \rho_* \pi_* (\pi^* M_0 \otimes \mathcal{L}(e_1 - s - f)) \cong \rho_* (M_0 \otimes \mathcal{L}(-s - f)), \end{aligned}$$

and thus $V'' \cong V \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$. We furthermore find that the corresponding map $\rho'_*(B'' \otimes \mathcal{L}(s' + f))$ on \mathbb{P}^1 factors as

$$V \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\rho_*(B \otimes \mathcal{L}(s))} W \otimes \rho_* \mathcal{L}(s - f) \xrightarrow{1 \otimes \psi} W \otimes \rho'_* \mathcal{L}(s') \quad (8.40)$$

where $\psi : \rho_* \mathcal{L}(s - f) \rightarrow \rho'_* \mathcal{L}(s')$ is the image of the natural map $\mathcal{L}(s - f) \rightarrow \mathcal{L}(s - e_1)$ on X_1 . But then, as a morphism on C , B'' is just the composition with the corresponding map of vector bundles on C . (If we want, we can then compute the true B' by restoring maximality.)

That is, if \mathcal{L}_0 is the original (degree 2) twisting bundle, then the new twisting bundle is $\mathcal{L}_1 := \mathcal{L}_0 \otimes \mathcal{L}(-p)$ where $p \in C$ is the point corresponding to e_1 ; and $B'' = \psi B$ where ψ is the unique (up to scalars) global section of

$$\mathcal{L}(-p) \otimes \pi_{\eta}^* \mathcal{O}_{\mathbb{P}^1}(1), \quad (8.41)$$

essentially a degree 1 theta function vanishing at $\eta(p)$. We then find that $A' = \eta^* \psi^{-1} \psi A$. In other words, the elementary transformation simply multiplies the shift matrix by a ratio of two degree 1 theta functions, preserving the symmetry.

The remaining simple reflection is much more subtle, as can be seen in particular by the fact that it does not preserve the rank of the equation. Indeed, if $D = ns + df - \sum_i r_i e_i$ is the original

class, then after reflecting in $s - f$, we obtain an equation of class $D = ds + nf - \sum_i r_i e_i$. Since this swaps the order of the equation and a measure of its degree (relative to ordinary multiplication), this suggests that this operation should correspond to some sort of generalized Fourier transformation. This is in fact the case, and the transform is essentially that of Spiridonov and Warnaar [54], but we postpone the discussion to [44], where the transform will play a crucial role.

As we mentioned above, we were led to consider symmetric elliptic difference equations by their appearance in two contexts: as the equations satisfied by elliptic hypergeometric integrals, and as equations related to elliptic biorthogonal functions (and Painlevé theory). We should therefore explain how these equations fit into the current framework.

In [47], Spiridonov and the author computed the explicit matrix A for the difference equation satisfied by the “order m elliptic beta integral”. For generic parameters, these equations are nonsingular at the ramification points, and it is thus straightforward to compute their singularity structure. The order m elliptic beta integral satisfies an elliptic equation of order $m + 1$ with $2m + 4$ “simple zeros”, i.e., points where A is holomorphic and $\det(A)$ vanishes once, as well as two points where A vanishes identically. This gives a sheaf of Chern class

$$(m + 1)s + (m + 2)f - (m + 1)e_1 - (m + 1)e_2 - \sum_{3 \leq i \leq 2m+6} e_i \quad (8.42)$$

on a blowup of F_2 (we can recover the coefficient of f by degree considerations once we have found all the singularities). If we perform an elementary transformation, swap e_1 and e_2 , then again perform an elementary transformation (i.e., if we reflect in $f - e_1 - e_2$), this gives us a sheaf on a blowup of F_0 or F_2 with Chern class

$$(m + 1)s + f - \sum_{3 \leq i \leq 2m+6} e_i, \quad (8.43)$$

reflecting the fact that the equation given in [47] had two singularities introduced precisely in order to make it elliptic rather than twisted. Since the first two blowups are independent of the remaining blowups, we can move those to the end, then ignore them. Thus the most natural sheaf-relaxation of this equation has Chern class

$$(m + 1)s + f - \sum_{1 \leq i \leq 2m+4} e_i \quad (8.44)$$

on a surface with $K_X^2 = 4 - 2m$, relative to an even blowdown structure. Now, this vector is actually a positive (real) root for E_{2m+5} , and thus (since by construction the sheaf is supported on the complement of the anticanonical curve) is generically the class of a -2 -curve. In particular, the sheaf, and thus the difference equation, is rigid; this of course explains why it was even possible to write down the equation explicitly.

To verify that the vector is a positive root, we can of course apply the usual algorithm. The only simple root that has negative intersection with the class is $s - f$, and thus the first step is the generalized Fourier transformation mentioned above. This gives a first-order equation of class

$$s + (m + 1)f - \sum_{1 \leq i \leq 2m+4} e_i, \quad (8.45)$$

at which point the action of D_{2m+4} suffices to transform it to the trivial equation $s - f$ (assuming sufficiently general parameters). Since first-order equations have explicit meromorphic solutions

given by elliptic Gamma functions [49], we see that the above $m + 1$ -st order equation should have a solution expressed as an integral involving elliptic Gamma functions. This is, of course, hardly surprising considering that the equation arose as the equation of an integral, and indeed, we recover the elliptic beta integral in this way.

Now, suppose one starts with the trivial equation and performs some sequence of elementary transformations in various points and Spiridonov-Warnaar transformations. This will have the effect of replacing the original -2 -curve $s - f$ by some image under $W(E_{m+1})$, which will still be the class of a -2 -curve, and thus corresponds to a rigid equation. (Indeed, for the relaxation, rigidity is a property of the sheaf on X , so is independent of the blowdown structure.) In terms of solutions, this starts with 1, and performs some sequence of the operations “multiply by a symmetric product of elliptic Gamma functions” and “apply the Spiridonov-Warnaar transform”. One thus expects that the result of such a sequence of operations will always satisfy a *rigid* difference equation. (Despite our identification of the reflection in $s - f$ with the Spiridonov-Warnaar transform, this is not quite a theorem; the action on difference equations is purely formal, and relies on an assumption that there are no extra residue terms coming from certain required contour shifts.) Conversely, since every rigid equation comes from a -2 -curve, the standard algorithm suggests a way of building up an integral representation for the solution to any rigid equation. (This appears related to the notion of Bailey chains/trees, see [52, 53] for the elliptic case.)

The other main motivating family of equations are those of [46], the equations satisfied by certain functions which are biorthogonal with respect to the order m elliptic beta integral. These equations are no longer explicit (though the residues can be expressed as multivariate integrals of products of elliptic Gamma functions), but it is still quite feasible to determine their singularities. These start out twisted, so we need to make one of the 16 compatible choices of twisting data, but there is a natural choice making the equation nonsingular at the ramification points, at least for generic parameters. (Alternately, as in the elliptic beta integral case, [46] gives a well-controlled elliptic version of the difference equation, corresponding to the non-elliptic version by a pair of elementary transformations.) We find that we are in the even case, and have a second-order equation with $2m + 6$ simple singularities. The corresponding Chern class is thus

$$2s + (m + 1)f - \sum_{1 \leq i \leq 2m+6} e_i. \quad (8.46)$$

When $m = 0$, this is rigid (and indeed all of the multivariate integrals arising as coefficients can be explicitly evaluated); this is of course one of the rigid cases we just saw, corresponding to the fact that the order 0 elliptic beta integral admits hypergeometric biorthogonal functions. (This generalizes the fact that the Jacobi polynomials, orthogonal with respect to the usual beta integral, are hypergeometric.) Otherwise, the class is in the fundamental chamber, so nef, and is easily checked to be generically integral. Since the coefficients of the Chern class are relatively prime, we find that the corresponding moduli space of sheaves is rational, and thus the same is true for the moduli space of difference equations (at least for the components where the moduli spaces are birational; of course, then q -deformed twisting should make this true for the difference equations in any component).

The case $m = 1$ is of particular interest, as in that case the 2-dimensional moduli space is an open subvariety of an elliptic surface, the relative Jacobian of the original X_8 (which is isomorphic to X_8 given the choice of a section). More precisely, it is obtained from the relative Jacobian by removing both the Jacobian of C_α and the divisor where the bundle fails to be trivial. As we mentioned, this is just the theta divisor, so corresponds to the identity section of the relative Jacobian. Since a section of a rational elliptic surface is a -1 -curve (it is smooth, rational, and

meets C_α in a single point), we could just as well blow down that section before removing it. This gives us an alternative interpretation of the moduli space as the complement of C_α in a del Pezzo surface of degree 1: the section blows down to a point of C_α , so there is nothing else to remove. The q -deformed twisting operations discussed above must still act as rational maps on this del Pezzo surface, and one can check directly that those actions all factor through blowing up the point of C_α corresponding to q [2]. In other words, the moduli space of difference equations of this type is precisely the sort of rational surface studied by [50], and the twist operations correspond directly to the elliptic Painlevé equation. Since the divisor class is anticanonical, its expansion in the standard basis is invariant under the action of $W(E_9)$ on blowdown structures, and thus we obtain an action of $W(E_9)$ on the family of moduli spaces of difference equations. The translations (which is how Sakai defined the elliptic Painlevé equation in [50]) act the same way on the parameters as the twist operations, and thus must actually act in the same way on the moduli spaces. In other words, there is a large abelian subgroup of $\mathbb{Z}^{10} \rtimes W(E_9)$ that acts trivially on the moduli space; this almost certainly is special to the $m = 1$ case. In particular, from a difference equation perspective, the correct way to generalize the elliptic Painlevé equation is clearly the twisting action rather than the Coxeter group action, as the former is what acts by gauge (i.e., isomonodromy) transformations in general. (In fact, the relation between these interpretations is mediated by a derived equivalence, see [44].)

For general m , the fact that B is a morphism between rank 2 bundles and becomes rank 1 at the singular points (at least generically) allows us to define a number of rational functions on the moduli space. A typical example is the following: when W is trivial, the 1-dimensional subspaces $\ell_i := \text{im}(B(p_i))$ determine eight points in a common projective line, and thus we can take the cross-ratio of any four of them to obtain a rational function on the moduli space. This function has a particularly nice divisor, and can formally be written as a ratio of tau functions. More precisely, if for any $v \in \mathbb{Z}^{2m+8}$, we define $\tau(v)$ to be the tau function which vanishes when $H^0(M \otimes \mathcal{L}(v-f)) \neq 0$, then

$$\chi(\ell_1, \ell_2, \ell_3, \ell_4) \propto \frac{\tau(f - e_1 - e_3)\tau(f - e_2 - e_4)}{\tau(f - e_1 - e_4)\tau(f - e_2 - e_3)}. \quad (8.47)$$

(This is really just a statement about the divisor of the left-hand side.) This follows from the observation that (for generic parameters) $\ell_i = \ell_j$ precisely when twisting by $-e_i - e_j$ changes W from $\mathcal{O}_{\mathbb{P}^1}^2$ to $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$, an easy consequence of our description of how twisting affects W . (It is not quite clear how this should extend to the locus $\tau(0)$ where W itself is not trivial, but it is clear that whatever order it has along $\tau(0)$ will not depend on i or j .) If we take into account the behavior when $\text{im}(B(p_i)) = \text{im}(B(p_j))$, this suggests a more precise statement

$$\chi(\ell_1, \ell_2, \ell_3, \ell_4) \propto^? \frac{\theta(e_1 - e_3)\tau(f - e_1 - e_3)\theta(e_2 - e_4)\tau(f - e_2 - e_4)}{\theta(e_1 - e_4)\tau(f - e_1 - e_4)\theta(e_2 - e_3)\tau(f - e_2 - e_3)}, \quad (8.48)$$

where for any positive root r , $\theta(r)$ is the divisor on the moduli stack of surfaces that vanishes where the root is a -2 -curve. This is very suggestive of the formula for the cross-ratio given in [46], in which there are multivariate integrals taking the places of factors

$$\frac{\tau(f - e_i - e_j)}{\theta(f - e_i - e_j)}; \quad (8.49)$$

and further suggests that we should have an equation of the form

$$\begin{aligned} & \theta(e_1 - e_2)\tau(f - e_1 - e_2)\theta(e_3 - e_4)\tau(f - e_3 - e_4) \\ & - \theta(e_1 - e_3)\tau(f - e_1 - e_3)\theta(e_2 - e_4)\tau(f - e_2 - e_4) \\ & + \theta(e_1 - e_4)\tau(f - e_1 - e_4)\theta(e_2 - e_3)\tau(f - e_2 - e_3) =^? 0, \end{aligned}$$

though it is as yet unclear how to make precise sense of such a statement. (The main issue is producing suitable canonical isomorphisms between tensor products of bundles of the form $\det \mathbf{R}\Gamma$, since θ and τ are canonical global sections of such bundles.) Note in particular that when $m = 1$, if we could make sense of the above relation, then it would induce an entire $W(E_9)$ -orbit of relations, which are precisely the relations satisfied by a tau function for the elliptic Painlevé equation, as given in [26, Thm. 5.2].

Remark. In [45], a four-term variant (related to Plücker relations for pfaffians) of this $W(E_9)$ -invariant system of recurrences was introduced, corresponding to a slightly different family of multivariate hypergeometric integrals. Is there a corresponding model (presumably involving relative Pryms rather than relative Jacobians) from a geometric perspective? This would presumably involve a component of the fixed locus of an involution of the form $M \mapsto \iota^* \mathcal{E}xt^1(M, \omega_X)$, where ι is an involution on an anticanonical surface of the form $X = X_8$ that acts as a hyperelliptic involution on the relevant anticanonical curve. (The latter condition ensures that the combined involution is Poisson.)

The case $D = 2rs + 2rf - \sum_{1 \leq i \leq 8} re_i$ is also likely to have interesting behavior (assuming it is generically integral, i.e., that $\mathcal{L}(\bar{C}_\alpha)|_{C_\alpha}$ has exact order r on X_8). In this case, X_8 is an elliptic surface on which C_α appears as the underlying curve of an r -fold section, but the moduli space is still an open subset of the relative Jacobian. As we saw, the moduli space of matrices B is always rational in the 2-dimensional case, and the same reasoning as before tells us that it is an affine del Pezzo surface of degree 1. Again, for any q , we have an induced action of $\mathbb{Z}^{10} \rtimes W(E_9)$ as birational maps on this family of del Pezzo surfaces. Since Theorem 7.1 tells us the parameters of these del Pezzo surfaces, we can control the action of the birational maps, and find that the action factors through a suitable one-point blowup, just as in the case $r = 1$. Note, however, that the lack of a universal family on this moduli space means that the most obvious way of producing a corresponding Lax pair will not work.

9 Degenerations

As one might expect, the story becomes more complicated once the anticanonical curve becomes singular. The simplest case is when the anticanonical curve on X is still integral; in that case, the considerations of the previous section carry over with little change. The main constraint is that the symmetric (ordinary and q -) difference equations must have only finite singularities, since blowing up the node or cusp will introduce a new component to the anticanonical curve. This can actually be violated in a mild way: if the difference equation is twisted by a line bundle, we can use the singularity to single out a global section of the bundle (modulo scalars), and in this way obtain an untwisted equation with only a mild singularity at ∞ . (E.g., in the q -difference case, the matrix A will no longer be 1 at ∞ , but will still be a multiple of the identity.) (Equations with more complicated singularities at ∞ might correspond to sheaves that cannot be separated from the anticanonical curve by a suitable blowup, though this can always be fixed by a finite number of “pseudo-twists” [42, Lem. 6.8], and we have seen that these correspond to gauge transformations.)

For more degenerate cases, we can still be guided by what happens in the elliptic case. For the $W(E_{m+1})$ action, the A_{m-1} subsystem merely permutes the singularities, while elementary transformations change the twisting bundle and multiply the shift matrix by a corresponding meromorphic section. The reflection in $s - f$, in contrast, can have a more significant effect on the nature of the equation.

We have already considered how the different equations look on F_2 , and something similar applies to F_0 or F_1 . Indeed, since elementary transformations should not affect the type of equation,

but can introduce or contract fibers which are components of C_α , the rule is quite simple: contract any components of class f , and then recognize the curve from the same list of possibilities as for F_2 . (More precisely, choose any section of the ruling which is transverse to C_α , and perform a sequence of elementary transformations moving that section to the -2 -curve of an F_2 ; the resulting anticanonical curve will be disjoint from C_α , and differs from the original only in self-intersections of components and the contraction of fibers.) Since this rule treats sections and fibers differently, it is clear that the result can depend on the choice of ruling on F_0 .

Moreover, it is in general not possible to avoid this issue. One might be tempted to adapt the algorithms for checking integrality to use only elements of the group that stabilizes the decomposition of C_α , but this encounters two significant problems: the group need not be a reflection group, and the reflection subgroup need not have finite rank. Either possibility denies us any kind of “fundamental chamber”; there need not be any computable fundamental domain for the action. One must thus use the full algorithm, and this can most certainly change the kind of equation.

As an example of how birational maps can change the type of an equation, consider the case of a nonsymmetric q -difference equation with three polar singularities. We recall that such equations (with even twisting; as mentioned, this allows $A(\infty)$ to be a general multiple of the identity) correspond to sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ with specified intersection with a union of two bilinear curves meeting in two distinct points. The constraint on singularities means that the sheaf meets the component of C_α corresponding to poles in the specified three points (and the restriction is a direct sum of structure sheaves of subschemes of the corresponding degree 3 scheme). Now, a bilinear curve in $\mathbb{P}^1 \times \mathbb{P}^1$ has self-intersection 2, so after blowing up three points, the strict transform has self-intersection -1 . The assumption on singularities means that this -1 -curve e is disjoint from the lift of M , and thus there are blowdown structures for which $c_1(M)$ is in the fundamental chamber and such that $e_m = e$. Such a blowdown structure blows down one of the two components of the anticanonical curve, so produces an integral anticanonical curve on the eventual Hirzebruch surface.

In other words, there is a birational map taking nonsymmetric q -difference equations with three polar singularities (and regular at 0 and ∞ , modulo twisting) to symmetric q -difference equations (again regular at 0 and ∞). By looking at what the algorithm does to move e to e_m , we see that this involves reflecting in $s - f$ precisely once. Before reflecting, the anticanonical curve on $\mathbb{P}^1 \times \mathbb{P}^1$ has two components with classes $s + 2f$ and s , while after reflecting it has components of classes $2s + f$ and f .

Remark. Of course, something similar applies if we have more than three polar singularities or the nonsymmetric equation is singular at 0 or ∞ ; the only difference is that the resulting symmetric equation will be singular at 0 and ∞ , possibly in a complicated way.

On F_0 , we have a total of 16 possible ways the anticanonical curve can decompose, each of which corresponds to a different kind of generalized Fourier transform. There are 10 such transforms that preserve the type of equation, and three pairs that change the type. Of those, one changes between symmetric and nonsymmetric q -difference equations, one is the ordinary difference analogue, and a final one changes between differential and ordinary difference equations (a Mellin/ z transform). For the transforms that preserve type, there are one each for the three integral types, as well as three transforms on nonsymmetric q -difference equations, two for ordinary difference equations, and two for differential equations. Of the latter, the most degenerate is just the Fourier/Laplace transform, while the other is essentially the transform used in [27] (usually called “middle convolution” in the later literature, though this is something of a misnomer).

As we mentioned above, we can also model certain Deligne-Simpson problems via moduli spaces of sheaves on rational surfaces, and this gives rise to additional interesting maps of moduli spaces.

One, of course, is the (essentially trivial) observation that moduli spaces of Fuchsian differential equations correspond to moduli spaces of solutions to additive Deligne-Simpson problems; in our terms, we can see this by noting that the anticanonical curves in the latter case become a double \mathbb{P}^1 once we contract all fiber components. There is another relation, though, which we consider in the multiplicative case. Recall that we modeled four-matrix Deligne-Simpson problems via sheaves with a quadrangular anticanonical curve in $\mathbb{P}^1 \times \mathbb{P}^1$ (with components of class f , f , s , and s). This is a somewhat cumbersome model for three-matrix problems, as we need to take one of the four matrices to be the identity. The corresponding component of the anticanonical curve contains a single singular point; if we blow up this point and blow down both the fiber and the section containing it, we obtain a sheaf on \mathbb{P}^2 . If g_1, g_2, g_3 are the three matrices with product 1, the sheaf on \mathbb{P}^2 is modeled by the cokernel of the matrix $x + g_1y + g_1g_2z$, and the conjugacy classes are determined by the restriction to the anticanonical curve $xyz = 0$. (The additive variant involves sheaves with specified restriction to $xy(x + y) = 0$.)

Much as in the case of a nonsymmetric difference equation with three poles, a three-matrix Deligne-Simpson problem in which one matrix has a quadratic minimal polynomial gives rise to a surface in which the anticanonical curve contains a -1 -curve disjoint from the relevant sheaf. In particular, we obtain a sheaf on an even Hirzebruch surface by blowing up the two roots of the minimal polynomial, then blowing down the anticanonical component. On that Hirzebruch surface, the anticanonical curve has two components, both of class $s + f$, and we thus obtain a nonsymmetric q -difference equation. (More precisely, we obtain such an equation *after* choosing one of the two rulings; this is tantamount to choosing one of the two roots of the minimal polynomial.)

This map can be made precise as follows. Let g_0, g_1, g_∞ be a solution to the Deligne-Simpson problem, with $(g_1 - 1)(g_1 - \beta^{-1}) = 0$. (We have rescaled the chosen root of the minimal polynomial to 1.) Define a matrix $A^+(z)$ with rational coefficients by

$$A^+(z) = (1 - g_\infty^{-1}z)^{-1}(1 - g_0z) = (g_\infty - z)^{-1}(g_\infty - g_1^{-1}z). \quad (9.1)$$

The matrix $A^+(z)$ fixes any vector fixed by g_1 , and thus has a well-defined action on the quotient $\text{im}(g_1 - 1)$. If $A(z)$ is the matrix of this action in some basis, then we find $A(0) = 1$, $A(\infty) = \beta$, so that A represents a twisted q -difference equation which is regular at 0 and ∞ . Moreover, we see that the zeros of A occur at the eigenvalues of g_0^{-1} , and the poles of A occur at the eigenvalues of g_∞ , and thus we obtain a rational map between the two moduli spaces. (It is unclear how to make the inverse map explicit, though it certainly exists, due to the description in terms of sheaves.) Note that if g_∞ has a cubic minimal polynomial, then we can proceed further, turning this q -difference equation with three polar singularities into a symmetric q -difference equation. Similarly, a solution to a three-matrix additive Deligne-Simpson problem with a quadratic minimal polynomial produces an ordinary difference equation, which can be further transformed to a symmetric equation if another minimal polynomial is cubic.

In [15], several natural multiplicative Deligne-Simpson problems were considered in which the moduli spaces were shown to be complements of anticanonical curves in del Pezzo surfaces. The present approach not only recovers these results, but gives an alternate intrinsic description of the del Pezzo surface, making it possible to identify the result explicitly. There were four problems considered there, one for each of the root systems of type D_4, E_6, E_7, E_8 ; we consider only the E_8 case in detail. In that case, the Deligne-Simpson problem is to classify $6l \times 6l$ matrices g_1, g_2, g_3 with $g_1g_2g_3 = 1$, such that g_1, g_2 , and g_3 have (specified) minimal polynomials of degrees 2, 3, and 6 respectively. The corresponding surface blows up \mathbb{P}^2 in the $2 + 3 + 6 = 11$ points corresponding to the roots of the minimal polynomial, and its anticanonical curve has decomposition

$$(h - e_1 - e_2) + (h - e_3 - e_4 - e_5) + (h - e_6 - e_7 - e_8 - e_9 - e_{10} - e_{11}). \quad (9.2)$$

The Chern class of the sheaf has the form

$$6lh - \sum_i r_i e_i, \quad (9.3)$$

where

$$r_1 + r_2 = r_3 + r_4 + r_5 = r_6 + r_7 + r_8 + r_9 + r_{10} + r_{11} = 6l. \quad (9.4)$$

The dimension of the corresponding moduli space is determined by the self-intersection of the divisor, which is maximized when $r_1 = r_2$, $r_3 = r_4 = r_5$, etc. This has self-intersection 0, so apart from some isolated -2 -curve cases, is the only interesting case. Now, $h - e_1 - e_2$ is a -1 -curve, so can be blown down, after which $h - e_3 - e_4 - e_5$ becomes a -1 -curve; after that, we end up on a 9-point blow up of \mathbb{P}^2 with an integral anticanonical curve, and our divisor class becomes a multiple of the anticanonical curve. Thus the solution will be generically irreducible, i.e., the divisor will be generically integral, precisely when

$$6h - 3e_1 - 3e_2 - 2e_3 - 2e_4 - 2e_5 - e_6 - e_7 - e_8 - e_9 - e_{10} - e_{11} \quad (9.5)$$

determines a line bundle on $xyz = 0$ of exact order l . (In other words, the corresponding product of zeros of the minimal polynomials must be an l -th root of unity.) In that case, we find that the relevant surface is elliptic, with an l -tuple fiber (of type I_1 in this case), and the moduli space is an open subset of the relative Jacobian.

Now, just as in the difference equation case, we have a simple numerical criterion for the sheaves in this open subset to have presentations involving trivial bundles: again, we want $H^0(M) = H^1(M) = 0$, and thus the moduli space is the complement of a fiber and section on the relative Jacobian. The fiber corresponds to the original l -tuple fiber, and has the same Kodaira type (since relative Jacobians preserve Kodaira types of tame multiple fibers, [10, Thm. 5.3.1]); since the anticanonical divisor on that surface was integral (nodal), we see that the fiber being removed is an integral nodal curve. Moreover, as in the elliptic case, we could blow down the section before removing it, and in this way obtain a del Pezzo surface of degree 1 with a nodal integral fiber removed. The problem of identifying this del Pezzo surface reduces to the problem of identifying the corresponding elliptic surface, and thus to a special case of Theorem 7.1. For $l = 1$, this agrees with the del Pezzo surface for which explicit equations were given in [15]; for $l > 1$, the conclusion of Theorem 7.1 settles the conjecture made there (to wit that the formula for the equation of the moduli space need only be modified by taking l -th powers of the input).

The E_7 and E_6 cases are similar: E_7 has $4l \times 4l$ matrices with minimal polynomials of degrees 2, 4, and 4, while E_6 has $3l \times 3l$ matrices with cubic minimal polynomials. In the E_6 case, the surface is already a relatively minimal elliptic surface (with an l -tuple fiber of type I_3), while in the E_7 case, we must blow down a component of the anticanonical curve, so end up with an l -tuple fiber of type I_2 . In the E_7 case, blowing down the tau divisor produces a -1 -curve in the anticanonical curve of the del Pezzo surface, so we can continue by blowing it down, and obtain a degree 2 del Pezzo surface with a nodal integral anticanonical curve removed. Similarly, in the E_6 case, we do this twice, and end up with an affine cubic surface with nodal curve at infinity. The D_4 case is somewhat different, in that it is a four-matrix Deligne-Simpson problem with quadratic minimal polynomials. We end up on a relatively minimal elliptic surface with a multiple I_4 fiber, but now have *two* tau divisors that need to be removed. After blowing down those -1 -curves, we have a degree 2 del Pezzo surface with a quadrangle at infinity, with two components of self-intersection -1 and two of self-intersection -2 . If we blow down one of the -1 -curve components, the result is a triangle of lines on a cubic surface; we could stop there (the description given for $l = 1$ in [36]), or continue to a degree 4 del Pezzo surface with a curve of type I_2 removed. (In this context, we

also note that [2] used explicit invariant theory to compute the moduli space of 2×2 matrices with specified determinant, where the entries are global sections of a degree 4 line bundle on a genus 1 curve; that the result is a del Pezzo surface of degree 2 follows in the same way, as again one must remove two tau divisors.)

In [11], multiplicative Deligne-Simpson problems were related to Coxeter groups, in this case to groups with arbitrary star-shaped Dynkin diagrams. (In particular, E_{m+1} has a star-shaped diagram, and the corresponding Deligne-Simpson problems have a quadratic and a cubic minimal polynomial.) At least in the three- and four-matrix cases, we can see these Coxeter groups from the rational surface perspective. In the three-matrix case, these are precisely the reflection subgroups of the stabilizer of the decomposition of C_α , see [33]; the simple roots of the subsystem are

$$h - e_{11} - e_{21} - e_{31}, \quad \text{and} \quad e_{ij} - e_{i(j+1)}, \quad (9.6)$$

where the first subscript on the e 's indicates which component of C_α they intersect. In the four-matrix case, the description is slightly more subtle: a four-matrix Deligne-Simpson problem corresponds to a sheaf which on the Hirzebruch surface has class $n(s+f)$; thus in addition to stabilizing the decomposition of C_α , we also want to stabilize the root $s-f$. In addition to the obvious A_i -subsystems, we have a simple root $s+f-e_{11}-e_{21}-e_{31}-e_{41}$. In any event, even if we assume that these reflections generate the full stabilizer (rather than just the subgroup generated by reflections), the algorithms still become more complicated. Indeed, we have already seen that there are several kinds of elliptic pencil on such a surface (even if we exclude multiple fibers), distinguished by the components that get blown down on the relatively minimal model; as a result, it is more difficult to identify whether a class is integral based merely on its image in the fundamental chamber of this smaller Coxeter group.

Of course, as we remarked above, the stabilizer of the anticanonical decomposition can fail to be a reflection group (and that reflection group can apparently fail to be of finite rank). There are a few cases [33] in which the stabilizer is a reflection group and has been explicitly identified, specifically those with nodal (and thus polygonal) anticanonical curve, having at most 5 components. (Looijenga also remarks that the 6 component case is probably feasible; and we have already seen that the stabilizer can fail to be a reflection group when there are 7 or more components.) The 2 component case, unsurprisingly, has a star-shaped diagram with one very short leg, corresponding to the relation between nonsymmetric q -difference equations and three-matrix multiplicative Deligne-Simpson problems in which one of the minimal polynomials is quadratic.

Just as the elliptic hypergeometric equation corresponds to a -2 -curve, so is rigid, most other hypergeometric difference/differential equations can be seen to be rigid in the same way. As an example, we consider a maximally degenerate case: the Airy function, which satisfies the non-Fuchsian differential equation

$$\text{Ai}''(z) = z \text{Ai}(z), \quad (9.7)$$

or in matrix form

$$v'(z) = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} v(z). \quad (9.8)$$

(Note that there is an ambiguity when passing between straight-line and matrix forms of an equation: the matrix form is only determined up to a gauge transformation, so (as long as we can avoid apparent singularities) the sheaf will only be determined up to “pseudo-twist”.) As we have seen, differential equations correspond to sheaves on F_2 with anticanonical curve of the form $y^2 = 0$; we

find that the above matrix translates to

$$\begin{pmatrix} w^2 & yx \\ y & w^3 \end{pmatrix} : \mathcal{O}_{F_2}(-s_{\min} - 2f) \oplus \mathcal{O}_{F_2}(-s_{\min} - 3f) \rightarrow \mathcal{O}_{F_2}^2. \quad (9.9)$$

In the q -difference case, we noted that we can often absorb particularly well-behaved singularities into a twist; something similar applies here, and we should perform an elementary transformation centered at the subscheme with ideal (y, w^2) . That is, blow up this subscheme, minimally desingularize, then blow down the original fiber and the -2 -curve. The resulting morphism on $\mathbb{P}^1 \times \mathbb{P}^1$ is

$$\begin{pmatrix} y_0 & y_1 x_1 \\ y_1 & y_0 x_0 \end{pmatrix}, \quad (9.10)$$

where the new coordinates relate to the original coordinates by

$$x_1/x_0 = x/w, \quad y_1/y_0 = y/w^2, \quad (9.11)$$

and the new anticanonical curve has equation $y_1^2 x_0^2$. The cokernel has smooth support, so there is no difficulty in resolving its intersection with the anticanonical curve. The support $y_0^2 x_0 = y_1^2 x_1$ meets the anticanonical curve in a 6-jet, and each of the first five blowups introduces a new component to the anticanonical curve, with multiplicities 3, 4, 3, 2, and 1 respectively. The result is a curve of Kodaira type III*, except with one reduced fiber removed. As for the sheaf itself, we started with support of class $2s + f$ and blew up 6 points, so the result is a -2 -curve as expected. (Had we started from the sheaf on F_2 , we would have obtained a sheaf with first Chern class class $2s + 3f - 2e_1 - 2e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8$, which is naturally also a positive (real) root.)

In general, a rigid second-order equation (of whatever kind) can always be transformed by a sequence of elementary transformations into one of class $2s + f - e_1 - e_2 - e_3 - e_4 - e_5 - e_6$. (That is, this is the unique class in the fundamental chamber with respect to D_m and satisfying $D \cdot f = 2$.) We can thus describe a moduli space of rigid equations, namely the locally closed substack of \mathcal{X}_6^α on which this class represents a -2 -curve. Since we can readily rule out the existence of $-d$ -curves for $d > 2$, it is a (reasonably small) finite problem to determine the different types of anticanonical surfaces that arise. There are a total of 3182 such types, but the action of $W(D_6)$ reduces this to only 41 equivalence classes, each of which describes a different kind of hypergeometric equation. These range from the elliptic hypergeometric equation (satisfied by the order 1 elliptic beta integral) down to the two maximally degenerate cases, the Airy equation and the q -difference equation $v(q^2 z) = \beta z v(z)$, passing through examples such as the differential and difference equations satisfied by the hypergeometric function of type ${}_2F_1$.

Similarly, in the case $2s + 2f - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8$, corresponding in the generic case to the elliptic Painlevé equation, which we saw above corresponded to 22 orbits of $W(E_8)$ (as in Figure 1 above), there are a total of 139981 types (we omit the figure!). The $W(E_8)$ orbits correspond to different types of Painlevé equation, but the Lax pairs themselves are essentially classified by the $W(D_8)$ orbits (since as we have seen, the map between equations and sheaves is only defined up to scalar gauge equivalence in any event). There are 61 such orbits, implying that a given (discrete or continuous) Painlevé equation can have qualitatively different second-order Lax pairs.

For instance, the classification indicates that the Painlevé VI equation should have a Lax pair in the form of a second order symmetric difference equation. Indeed, there is a $W(D_8)$ -orbit of types of surfaces in which the anticanonical curve decomposes as

$$(2s + f - e_1 - e_2 - e_3 - e_4 - e_5 - e_6) + (f - e_5 - e_6) + (e_5 - e_6) + 2(e_6 - e_7) + (e_7 - e_8), \quad (9.12)$$

which by our discussion of singularities above corresponds to a symmetric difference equation with four finite singularities and an indecomposable singularity at infinity. The corresponding Lax pair was constructed in [39], along with degenerations to Painlevé V and III.

The case $r(2s+2f-e_1-\cdots-e_8)+e_8-e_9$, which corresponds to an elliptic version of the “matrix Painlevé equation” [28], has precisely the same classification (apart from blowing up a point of an anticanonical component of self-intersection -1); there are two apparent additional types for $r = 2$, but they have a component $e_8 - e_9$, preventing the linear system from being generically integral.

The case $2s + 3f - e_1 - \cdots - e_{10}$, which generically corresponds to the simplest non-Painlevé case of the elliptic Garnier equation [38], gives rise to a total of 6374578 types in 84 $W(D_{10})$ -orbits. Note that since the divisor class is $-K + f$, it is not preserved by reflection in $s - f$; as a result, each degeneration of the simplest elliptic Garnier equation corresponds to a unique type of second-order Lax pair up to scalar gauge equivalence. (There are, of course, higher order Lax pairs, e.g., the third-order Lax pair obtained by reflecting in $s - f$.)

10 Calogero-Moser spaces

In [13, Defn. 5.2.3], Etingof, Gan, and Oblomkov considered a family of four multiplicative Deligne-Simpson problems associated to representations of a certain family of algebras, such that the associated moduli space of solutions could be interpreted as generalized Calogero-Moser spaces. (To be precise, the moduli space of solutions of the analogous additive Deligne-Simpson problems can be identified with the spherical subalgebra of a certain generalized double affine Hecke algebra, per [14, 13].) As above, these moduli spaces can be identified with moduli spaces of sheaves on rational surfaces of the form considered in Section 9. We find that with a suitable choice of blowdown structure, the corresponding divisor class is $D = -nK_8 + e_8 - e_9$. Thus more generally, we should expect suitable open subsets of moduli spaces $\mathcal{Irr}_X(D, x)$ to play a similar role.

Now, this same divisor class appeared in Theorem 6.1 as one of the two cases in which a nef divisor disjoint from C_α could fail to have a generically integral linear system. In particular, in the codimension 1 substack of $\mathcal{X}_9^{\alpha, \leq 2}$ such that $\mathcal{L}(-nK_8 + e_8 - e_9)_{C_\alpha} \cong \mathcal{O}_{C_\alpha}$, there is a smaller substack (of codimension 2 in $\mathcal{X}_9^{\alpha, \leq 2}$) on which the general section of the linear system $|-nK_8 + e_8 - e_9|$ is reducible. We will see that, at least generically, the corresponding symplectic moduli space of sheaves can be identified with the Hilbert scheme of n points on a suitable quasiprojective surface, and thus the full family of moduli spaces provides flat deformations of such moduli spaces.

Since $e_9 \cdot D = 1$, we may feel free to consider instead the divisor class $-nK_8 + e_8$ on the surface $X = X_8$, as the remaining point of intersection is then uniquely determined. The generically reducible case has $|-K_8|$ a pencil, and thus X is in this case an elliptic surface with a section.

It will be convenient to first consider the other generically reducible case corresponding to elliptic surfaces.

Lemma 10.1. *Let $\psi : X \rightarrow C$ be a smooth, relatively minimal, genus 1 fibration with no multiple fibers, and let $\mathcal{M}_X^{(n)}(0)$ be the moduli space classifying stable sheaves M of Euler characteristic 0 and with $c_1(M)$ a sum of n fibers of ψ . Then $\mathcal{M}_X^{(n)}(0)$ is naturally isomorphic to $\mathrm{Sym}^n(W)$, where W is the Weierstrass model of the relative Jacobian of X .*

Proof. Any sheaf M classified by $\mathcal{M}_X^{(n)}(0)$ is S-equivalent to a sum of stable sheaves. Since a stable sheaf has connected support, we find that each summand is set-theoretically supported on a single fiber, allowing us to apply Lemma 7.5. We thus find that M is S-equivalent to a sum of (a) invertible sheaves on fibers, (b) torsion-free but not integral sheaves on integral fibers, and (c) images of $\mathcal{O}_{\mathbb{P}^1}(-1)$ under maps to reducible fibers. The first two cases have the same first Chern

class as a fiber, while the multiplicities of the summands of the third type are uniquely determined by their contribution to the first Chern class. In particular, we find that M is S-equivalent to a sum of n semistable sheaves, each of which has c_1 the class of a single fiber; the summands are further uniquely determined up to S-equivalence. That $\mathcal{M}_X^{(n)}(0)$ is the Weierstrass model of the relative Jacobian then follows from Corollary 7.6. \square

Theorem 10.2. *Let $\psi : X \rightarrow \mathbb{P}^1$ be a smooth, relatively minimal, genus 1 fibration with no multiple fibers and a section s , and let f denote the class of a fiber of ψ . Then the moduli space $\mathcal{M}_X^{(n)}(1)$ classifying stable sheaves M of Euler characteristic 1 with $c_1(M) = nf + s$ is a locally symplectic resolution of $\text{Sym}^n(W)$, where W is the Weierstrass model of X .*

Proof. A divisor D linearly equivalent to $nf + s$ meets the generic fiber transversely, and in the same point as s (since $\mathcal{L}(D)$ meets the generic fiber in the same line bundle as $\mathcal{L}(s)$). It in particular follows that s is a component of D (and the unique horizontal component). In particular, if M is any sheaf corresponding to a point of $\mathcal{M}_X^{(n)}(1)$, the support of M contains s .

More generally, let M be a stable sheaf of Euler characteristic 1 such that the 0-th Fitting scheme is a sum of s and a nonnegative linear combination of components of fibers. We claim that M admits a surjective morphism to \mathcal{O}_s . If M is supported on s , this is immediate. Otherwise, let C be a fiber containing some vertical component of the support of M , and consider the image M' of the natural morphism $M \rightarrow M \otimes \mathcal{L}(C)$. Now, M' is a (proper) nonzero quotient of M , so $\chi(M') > 0$, while $M' \otimes \mathcal{L}(-C)$ is a proper subsheaf of M , so $\chi(M' \otimes \mathcal{L}(-C)) \leq 0$. Since $c_1(M) - c_1(M')$ is vertical, we find that $c_1(M') \cdot C = c_1(M) \cdot C = 1$, and thus $\chi(M') = \chi(M' \otimes \mathcal{L}(-C)) + 1$, implying $\chi(M') = 1$. Since any quotient of M' is a quotient of M , we see that M' is again stable, so by induction admits a surjection to \mathcal{O}_s , giving the required surjection $M \rightarrow \mathcal{O}_s$.

Now, let M be the sheaf corresponding to a point of $\mathcal{M}_X^{(n)}(1)$. Then M certainly has a surjection to \mathcal{O}_s , and since the kernel has no map to \mathcal{O}_s , this surjection is unique. Thus M uniquely determines the kernel M_v of this surjection, a sheaf of Euler characteristic 0 with $c_1(M_v) = nf$ such that any subsheaf (being a proper subsheaf of M) has nonpositive Euler characteristic. In particular, M_v is semistable, so that we obtain a morphism from $\mathcal{M}_X^{(n)}(1)$ to the corresponding semistable moduli space, which by Lemma 10.1 is isomorphic to $\text{Sym}^n(W)$.

It remains only to show that this map is locally (on the base) a symplectic resolution. If X is symplectic, then $\mathcal{M}_X^{(n)}(1)$ is symplectic, and this is immediate. More generally, if X is Poisson, then so is $\mathcal{M}_X^{(n)}(1)$, and moreover $\mathcal{M}_X^{(n)}(1)$ is generically symplectic; thus the symplectic locus is a symplectic resolution of the symmetric power of the Weierstrass model of the symplectic locus of X . By varying the Poisson structure, we can cover $\text{Sym}^n(W)$ by open subvarieties over which $\mathcal{M}_X^{(n)}(1)$ is a symplectic resolution.

The remaining case is that $-K_X$ is ineffective, but then K_X is a positive sum of fibers, making K_X effective. Any nonzero section β of ω_X induces a (closed) 2-form on $\mathcal{M}_X^{(n)}(1)$ (see, e.g., [24, §10]). Moreover, we claim that this 2-form is nondegenerate whenever the corresponding sheaf is transverse to the zero locus of β ; again, by varying β , this will suffice to make $\mathcal{M}_X^{(n)}(1) \rightarrow \text{Sym}^n(W)$ a symplectic resolution locally on the base.

We need to show that the induced (and self-dual) morphism $\text{Ext}^1(M, M) \rightarrow \text{Ext}^1(M, M \otimes \omega_X)$ is an isomorphism, or equivalently that the connecting map $\text{Hom}(M, M \otimes \text{coker}(\beta)) \rightarrow \text{Ext}^1(M, M)$ is 0. Now, $M \otimes \text{coker}(\beta)$ is a 0-dimensional quotient of \mathcal{O}_s of degree $d - 2$, so that

$$\dim \text{Hom}(M, M \otimes \omega_X) \leq d - 1, \quad (10.1)$$

with equality iff the connecting map vanishes. Since ω_X has this many global sections, none of which vanish on the support of M , the claim follows. \square

Over a field of characteristic 0, the main result of [16] shows that for any smooth surface X , the natural morphism $\mathrm{Hilb}^n(X) \rightarrow \mathrm{Sym}^n(X)$ from the Hilbert scheme of points is the unique crepant resolution of $\mathrm{Sym}^n(X)$. In particular, when W is smooth (and characteristic 0), then $\mathcal{M}_X^{(n)}(1) \cong \mathrm{Hilb}^n(W)$. When W is not smooth, the most natural choice of crepant resolution of $\mathrm{Sym}^n(W)$ is the Hilbert scheme of its minimal desingularization X . However, it turns out [17] that this is not the crepant resolution given by $\mathcal{M}_X^{(n)}(1)$, which is in fact given by the “perverse” Hilbert scheme of W .

In the rational case, we may more generally consider the moduli spaces $\mathcal{M}_X(-nK_8 + e_8, 1)$ classifying semistable 1-dimensional sheaves of Euler characteristic 1 and first Chern class $-nK_8 + e_8$, which we can view as a family of deformations of (perverse) Hilbert schemes of points on (Jacobian) rational elliptic surfaces. Replacing X by a noncommutative deformation should give an additional parameter in the family of moduli spaces, which should afford a deformation of the (perverse, presumably) Hilbert scheme of points on a general rational surface with $K_X^2 = 0$. There is also a notion of a Hilbert scheme of a noncommutative surface [35, 44], presumably related to this case by a derived equivalence. Another flat deformation of the symmetric power of a general rational surface will be constructed in [43] as a special case of a noncommutative deformation of such symmetric powers.

We can also obtain deformations of (perverse) Hilbert schemes of affine del Pezzo surfaces (i.e., the complement of an anticanonical curve on such a surface). Indeed, for X elliptic such that $X \setminus C_\alpha$ contains no -2 -curve, we have already seen that the symplectic leaf of $\mathcal{M}_X(-nK_8 + e_8, 1)$ is a symplectic resolution of $\mathrm{Sym}^n(X \setminus C_\alpha)$, so in general we obtain a flat family of symplectic varieties having an additional parameter. To obtain a corresponding deformation for del Pezzo surfaces, we need simply remove a suitable combination of tau divisors. (In particular, this allows us to identify the Deligne-Simpson moduli spaces considered above with symplectic deformations of symmetric powers of the obvious del Pezzo surfaces.)

Similar considerations apply in the case $X = E \times \mathbb{P}^1$, as again the relevant divisor class remains effective and generically disjoint from the anticanonical curve for a larger family of moduli problems.³ These moduli problems have been considered before, see [22, §10.1]. In particular, when X is the \mathbb{P}^1 -bundle associated to an indecomposable bundle of rank 2 and trivial determinant, the corresponding moduli space is the ambient space of the elliptic Calogero-Moser system of type GL_n .

With this in mind, we refer to the action of the relevant lattice on the above rational moduli spaces as (degenerations of) the “symmetric elliptic difference Calogero-Moser system”. (Note that the usual action of Λ_{E_8} is extended by an additional twist, since we have an additional point to blow up.) It would be particularly interesting to understand the noncommutative deformation of this action, as it would give a 1-parameter deformation of the n -th symmetric power of the elliptic Painlevé equation (and of the other equations in Sakai’s hierarchy), the differential level of which has recently been constructed [28].

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³There is one more case with this property: for each $g \geq 2$, the moduli stack of K3 surfaces of genus g contains (as a codimension 1 substack) the moduli stack of Jacobian elliptic K3 surfaces polarized by $gs + f$.

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